

INSTITUTE OF ACTUARIES OF INDIA

CS2A - Risk Modelling and Survival Analysis

May 2026 Examination

INDICATIVE SOLUTION

Introduction:

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiners have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1	C	[2 Marks]
Solution 2	A	[2 Marks]
Solution 3	C	[2 Marks]
Solution 4	B	[2 Marks]
Solution 5	A	[2 Marks]
Solution 6	D	[2 Marks]
Solution 7	D	[2 Marks]
Solution 8	D	[2 Marks]
Solution 9	C	[2 Marks]
Solution 10	C	[2 Marks]
Solution 11	A	[2 Marks]
Solution 12	D	[2 Marks]
Solution 13	B	[2 Marks]
Solution 14	C	[2 Marks]
Solution 15	A	[2 Marks]

Solution.16

- i) Strictly stationary processes -the distribution of $(X_{t+1}, \dots, X_{t+k})$ is the same as that of $(X_{t+s+1}, \dots, X_{t+s+k})$ for each t, s and k .

Joint distributions are identical over time $\rightarrow$ the process remains same as time elapse

Weakly stationary process - only the first two moments are needed to satisfy the conditions,

$$E(X_t) = \mu, \text{ for all } t,$$

$$\text{cov}(X_t, X_{t+s}) = \gamma(s), \text{ for all } t, s.$$

the process has constant variance and variance is independent of time t [1]

- ii) Both Strictly stationary and Weakly stationary processes coincide for the multivariate normal processes

As normal distribution is defined by the first two moments only. [1]

- iii) Inputs given are:

Sample size is 250

$$r_1 = 0.042, r_2 = 0.191, r_3 = 0.023, r_4 = -0.085, r_5 = -0.021, r_6 = 0.032$$

Testing of Null hypothesis:

H0: The sample residual values conform to white noise process

H1: The sample residual values don't conform to white noise process

$$n = 250 \text{ \& } m = 6$$

The test statistics value is, $n(n+2) \sum_{k=1}^m (r_k^2 / (n-k))$ (0.5)

$$= 250 * 252 * \left(\frac{0.042^2}{249} + \frac{0.191^2}{248} + \frac{0.023^2}{247} + \frac{(-0.085)^2}{246} + \frac{(-0.021)^2}{245} + \frac{(0.032)^2}{244} \right)$$

$$= 12.077$$

Test statistics value of χ_6^2 distribution at 5% level is, 12.592

The above test concludes that there are no adequate evidence to reject the null hypothesis and confirms residual values follow white noise process

[3]

iv) Given, $X_t = a_1X_{t-1} + a_2X_{t-2} + b_1e_{t-1} + e_t$

For this model,

$$\text{Cov}(X_t, e_t) = \sigma^2 \text{ and}$$

$$\text{Cov}(X_t, e_{t-1}) = a_1\text{Cov}(X_{t-1}, e_{t-1}) + b_1\sigma^2 = (a_1 + b_1)\sigma^2$$

Taking the covariance of the defining equation with X_t we get:

$$\gamma_0 = a_1\gamma_1 + a_2\gamma_2 + b_1(a_1 + b_1)\sigma^2 + \sigma^2$$

$$\text{Taking the covariance with } X_{t-1} \text{ we get: } \gamma_1 = a_1\gamma_0 + a_2\gamma_1 + b_1\sigma^2$$

$$\text{Taking the covariance with } X_{t-2} \text{ we get: } \gamma_2 = a_1\gamma_1 + a_2\gamma_0$$

As a result of this, in general the equation takes the form of, $\gamma_n = a_1\gamma_{n-1} + a_2\gamma_{n-2}$ for $n > 2$

[4]

v) Derive value for ρ_k

The autocovariance at lag 1 is, $\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$

$$\gamma_1 = \text{cov}(X_t, X_{t-1}) = \text{cov}\left(\frac{9}{20}X_{t-1} - \frac{1}{20}X_{t-2} + e_t, X_{t-1}\right)$$

$$\gamma_1 = \frac{9}{20}\text{cov}(X_{t-1}, X_{t-1}) - \frac{1}{20}\text{cov}(X_{t-2}, X_{t-1}) + \text{cov}(e_t, X_{t-1})$$

$$\gamma_1 = \frac{9}{20}\gamma_0 - \frac{1}{20}\gamma_1$$

$$\frac{21}{20}\gamma_1 = \frac{9}{20}\gamma_0$$

$$\gamma_1 = \frac{3}{7}\gamma_0$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{3}{7}$$

Similarly, the autocovariance at lag 2 is,

$$\gamma_2 = \text{cov}(X_t, X_{t-2}) = \text{cov}\left(\frac{9}{20}X_{t-1} - \frac{1}{20}X_{t-2} + e_t, X_{t-2}\right)$$

$$\gamma_2 = \frac{9}{20}\text{cov}(X_{t-1}, X_{t-2}) - \frac{1}{20}\text{cov}(X_{t-2}, X_{t-2}) + \text{cov}(e_t, X_{t-2})$$

$$\gamma_2 = \frac{9}{20}\gamma_1 - \frac{1}{20}\gamma_0$$

$$\gamma_2 = \frac{9}{20} \cdot \frac{3}{7}\gamma_0 - \frac{1}{20}\gamma_0$$

$$\gamma_2 = \frac{27}{20 \cdot 7}\gamma_0 - \frac{1}{20}\gamma_0 = \frac{1}{7}\gamma_0$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{1}{7}$$

In general, for $k \geq 2$, we have:

$$\rho_k = \frac{9}{20}\rho_{k-1} - \frac{1}{20}\rho_{k-2}$$

$$\lambda^2 - \frac{9}{20}\lambda + \frac{1}{20} = (\lambda - \frac{1}{4})(\lambda - \frac{1}{5})$$

$$\rho_k = \left(\frac{1}{4}\right)^k A + \left(\frac{1}{5}\right)^k B$$

$$\rho_0 = A + B = 1$$

$$\rho_1 = \left(\frac{1}{4}\right)A + \left(\frac{1}{5}\right)B = \frac{3}{7}$$

$$A = \frac{32}{7}, B = -\frac{25}{7}$$

$$\rho_k = \frac{32}{7}\left(\frac{1}{4}\right)^k - \frac{25}{7}\left(\frac{1}{5}\right)^k$$

[5]

[14 Marks]

Solution.17

i)

a) n_{ij} , values:

The results are tracked among the two consecutive rounds to analyse the movements.

n11	3
n12	4
n13	3
n21	5
n22	2
n23	1
n31	1
n32	3
n33	1

$$n_{ij} = \begin{pmatrix} 3 & 4 & 3 \\ 5 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

[2]

b) n_i , values: the value of it comes from n_{ij} matrix, sum of each row total, forms n_i values

$$n_i = \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix}$$

[1]

ii) Estimation of one-step probabilities among 3X3 matrix, calculated using formula of

$$\hat{p}_{ij} = \frac{n_{ij}}{n_i}$$

$$\begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.625 & 0.250 & 0.125 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

[2]

[5 Marks]

Solution.18

i) A healthy male life aged 30 can die over one year by following any one of the following four paths:

- Transition directly from Healthy(H) to Death(D) within one year (H-D)
- Transition from Healthy to Mental issues (Mi) followed by transition from Mi to D in one year (H-Mi-D)
- Transition from Healthy to Metabolic (Me) followed by transition from Me to D in one year (H-Me-D)
- Transition from Healthy to Accident/rest (Ar) followed by transition from Ar to D in one year (H-Ar-D)

H	Mi	Me	Ar	D
-0.088	0.032	0.021	0.016	0.019
	-0.6			0.6
		-0.8		0.8
			-0.4	0.4

[2]

ii) Probability of male aged 30 healthy to die in one year period

Healthy to dead directly:

$P_{H-D} = \int_{t=0}^1 P_{H-H}(t) \mu_{H-D} dt$, i.e., the probability of remain healthy until time t, then dying from healthy at that point, integrated over all possible times t within the one year ($0 < t < 1$).

$P_{H-H}(t) = P_{\overline{H-H}}(t) = e^{-(\mu_{H-D} + \mu_{H-Mi} + \mu_{H-Me} + \mu_{H-Ar})t} = e^{-0.088t} \rightarrow$ probability to be remain healthy till time t

$$\begin{aligned} \text{So, } P_{H-D} &= \int_{t=0}^1 e^{-0.088t} * 0.019 dt = 0.019 * \left[\frac{e^{-0.088t}}{-0.088} \right]_0^1 \\ &= \frac{0.019}{0.088} (1 - e^{-0.088}) = \mathbf{0.018188} \end{aligned}$$

Healthy to dead which had Mental issues:

$P_{H-Mi-D} = \int_{t=0}^1 P_{H-H}(t) \mu_{H-Mi} P_{Mi-D}(1-t) dt = \mu_{H-Mi} \int_{t=0}^1 P_{H-H}(t) P_{Mi-D}(1-t) dt$,
i.e., the probability of staying healthy until time t, contracting disease “Mi” at that point, and then dying at some point in the remaining (1-t) of the year.

$P_{Mi-D}(1-t) = 1 - P_{Mi-Mi}(1-t) = 1 - e^{-0.6(1-t)}$, as it is balance of probability of not remain at same state.

$$\begin{aligned}
&\text{This leads to, } P_{H-Mi-D} = 0.032 \int_{t=0}^1 e^{-0.088t} (1 - e^{-0.6(1-t)}) dt \\
&= 0.032 \left(\int_{t=0}^1 e^{-0.088t} dt - e^{-0.6} \int_{t=0}^1 e^{-0.512t} dt \right) \\
&= 0.032 \left[\frac{1 - e^{-0.088}}{0.088} - e^{-0.6} \left[\frac{e^{0.512} - 1}{0.512} \right] \right] \\
&= 0.032(0.957263 - 0.716698) = 0.032 * 0.240565 = \mathbf{0.007698}
\end{aligned}$$

Healthy to death which had Metabolic issues: similarly,

$P_{H-Me-D} = \int_{t=0}^1 P_{H-H}(t) \mu_{H-Me} P_{Me-D}(1-t) dt = \mu_{H-Me} \int_{t=0}^1 P_{H-H}(t) P_{Me-D}(1-t) dt$,
i.e., the probability of staying healthy until time t, contracting disease “Me” at that point, and then dying at some point in the remaining (1-t) of the year.

$P_{Me-D}(1-t) = 1 - P_{Me-Me}(1-t) = 1 - e^{-0.8(1-t)}$, as it is balance of probability of not remain at same state.

$$\begin{aligned}
&\text{This leads to, } P_{H-Me-D} = 0.021 \int_{t=0}^1 e^{-0.088t} (1 - e^{-0.8(1-t)}) dt \\
&= 0.021 \left(\int_{t=0}^1 e^{-0.088t} dt - e^{-0.8} \int_{t=0}^1 e^{-0.712t} dt \right) \\
&= 0.021 \left[\frac{1 - e^{-0.088}}{0.088} - e^{-0.8} \left[\frac{e^{0.712} - 1}{0.712} \right] \right] = 0.021(0.957263 - 0.655101) \\
&= 0.021 * 0.302162 = \mathbf{0.006345}
\end{aligned}$$

Healthy to death which had Accident/rest issues: similarly,

$P_{H-Ar-D} = \int_{t=0}^1 P_{H-H}(t) \mu_{H-Ar} P_{Ar-D}(1-t) dt = \mu_{H-Ar} \int_{t=0}^1 P_{H-H}(t) P_{Ar-D}(1-t) dt$,
i.e., the probability of staying healthy until time t, contracting disease “Ar” at that point, and then dying at some point in the remaining (1-t) of the year.

$P_{Ar-D}(1-t) = 1 - P_{Ar-Ar}(1-t) = 1 - e^{-0.4(1-t)}$, as it is balance of probability of not remain at same state.

$$\begin{aligned}
&\text{This leads to, } P_{H-Ar-D} = 0.016 \int_{t=0}^1 e^{-0.088t} (1 - e^{-0.4(1-t)}) dt \\
&= 0.016 \left(\int_{t=0}^1 e^{-0.088t} dt - e^{-0.4} \int_{t=0}^1 e^{-0.312t} dt \right) \\
&= 0.016 \left[\frac{1 - e^{-0.088}}{0.088} - e^{-0.4} \left[\frac{e^{0.312} - 1}{0.312} \right] \right] \\
&= 0.016(0.957263 - 0.786669) = 0.016 * 0.170593 = \mathbf{0.002729}
\end{aligned}$$

So the total probability of dying is:

$$\begin{aligned}
q_{30} &= P_{H-D} + P_{H-Mi-D} + P_{H-Me-D} + P_{H-Ar-D} \\
&= 0.018188 + 0.007698 + 0.006345 + 0.002729 = \mathbf{0.034961}
\end{aligned}$$

[9]

iii) The impact analysis:

State	Transition probability before using drug for each state	Transition probability after using drug for each state	Impact on probability
Healthy to dead directly	0.018188	0.018188	0.0%
Healthy to dead which had Mental issues	0.007698	0.004975	-35.4%
Healthy to death which had Metabolic issues	0.006345	0.002939	-53.7%
Healthy to death which had Accident/rest issues	0.002729	0.002729	0.0%
Total	0.034961	0.028832	-17.5%

Comments:

- The change in transition rates are in line with the corresponding reduction in transition rates for the two specific states
- The change in probabilities are lesser than change in the transition rates for those two specific states
- Where there are no change in transition rates from a given state, there are no impacts on overall transition rates
- Overall impact of 17.5% depend on the collective transition rate from each state, looks reasonable

[3]**[14 Marks]****Solution.19**

i) The proportional hazard function is,

$$\lambda(t; Z_i) = \lambda_0(t) * \exp(-0.205 * Z_1 + 0.104 * Z_1 + 0.391 * Z_2 + 0.459 * Z_3 - 0.046 * Z_4 + 0.378 * Z_5)$$

$\lambda(t; Z_i)$ – the hazard function for the i th life, at time t & $\lambda_0(t)$ – baseline hazard

Covariate Z_1 – 1, if insured is smoker and 0 for non-smoker

Covariate Z_2 – 1, if insured regular in diet & exercise, and 0 for irregular with it

Covariate Z_3 – 1, if insured regularly eats from outside, no exercise, 0 otherwise

Covariate Z_4 – 1, if insured is non-alcoholic and sleeps for 8 hours and 0 for alcoholic and sleeps less

Covariate Z_5 – 1, if insured is a heavy drinker and works in night shift and 0 otherwise

[4]

ii) Probability of 80% to recover from illness within two days of hospital stay, if a person who is non-smoker, regular on diet & exercise, non-alcoholic, sleeps for 8 hours.

$$\exp(\int_0^2 -\lambda_0(t) * \exp(-0.025 - 0 - 0.046) dt) = 0.80$$

$$\exp(\int_0^2 -\lambda_0(t) * \exp(-0.071) dt) = 0.80$$

$$\exp(\int_0^2 -\lambda_0(t) dt) = 0.80^{\exp(0.071)} = 0.786972 \rightarrow \text{equation 1}$$

- a. Smoker - regularly eats from outside, no exercise - Heavy drinking, Night shift

Probability of recovering from illness within two days of hospital stay:

$$\begin{aligned}
 &= \exp\left(\int_0^2 -\lambda_0(t) * \exp(0.104 + 0.459 + 0.378) dt\right) \\
 &= \exp\left(-\int_0^2 \lambda_0(t) dt * \exp(0.941)\right) = \exp\left(-\int_0^2 \lambda_0(t) dt\right)^{\exp(0.941)} \\
 &= 0.786972^{2.562543} \quad \text{using result of equation 1} \\
 &= 0.541241
 \end{aligned}$$

So, probability of required length of stay in hospital for more than two days
 $= 1 - 0.541241 = 0.458759$

- b. Non-smoker, irregular on diet & exercise, but non-alcoholic, sleeps 8 hours

Probability of recovering from illness within two days of hospital stay:

$$\begin{aligned}
 &= \exp\left(\int_0^2 -\lambda_0(t) * \exp(-0.025 + 0.391 + 0 - 0.046) dt\right) \\
 &= \exp\left(\int_0^2 -\lambda_0(t) dt * \exp(0.32)\right) \\
 &= 0.786972^{1.377128} \quad \text{using result of equation 1} \\
 &= 0.71899
 \end{aligned}$$

So, probability of required length of stay in hospital for more than two days
 $= 1 - 0.71899 = 0.28101$

[4]

- iii) With 10% increase in probability, the revised probability is 88% to recover from illness within two days of hospital stay, if a person who is non-smoker, regular on diet & exercise, non-alcoholic, sleeps for 8 hours, aged below 30.

$$\exp\left(\int_0^2 -\lambda_0(t) * \exp(-0.025 - 0 - 0 - 0.046 - 0.053) dt\right) = 0.88$$

$$\exp\left(\int_0^2 -\lambda_0(t) * \exp(-0.124) dt\right) = 0.88$$

$$\exp\left(\int_0^2 -\lambda_0(t) dt\right) = 0.88^{\exp(0.124)} = 0.865274 \rightarrow \text{equation 1}$$

- c. Smoker - regularly eats from outside, no exercise - Heavy drinking, Night shift

Probability of recovering from illness within two days of hospital stay:

$$\begin{aligned}
 &= \exp\left(\int_0^2 -\lambda_0(t) * \exp(0.104 + 0.234 + 0.225 + 0.378 - 0.053) dt\right) \\
 &= \exp\left(-\int_0^2 \lambda_0(t) dt * \exp(0.888)\right) = \exp\left(-\int_0^2 \lambda_0(t) dt\right)^{\exp(0.888)} \\
 &= 0.865274^{2.430264} \quad \text{using result of equation 1} \\
 &= 0.703504
 \end{aligned}$$

So, probability of required length of stay in hospital for more than two days

$$= 1 - 0.703504 = 0.296496$$

- d. Non-smoker, irregular on diet & exercise, but non-alcoholic, sleeps 8 hours

Probability of recovering from illness within two days of hospital stay:

$$\begin{aligned}
&= \exp\left(\int_0^2 -\lambda_0(t) * \exp(-0.025 + 0.198 + 0.193 - 0.046 - 0.053) dt\right) \\
&= \exp\left(-\int_0^2 \lambda_0(t) dt * \exp(0.267)\right) = \exp\left(-\int_0^2 \lambda_0(t) dt\right)^{\exp(0.267)} \\
&= 0.865274^{1.30604} \text{ using result of equation 1} \\
&= 0.82779
\end{aligned}$$

So, probability of required length of stay in hospital for more than two days
 $= 1 - 0.82779 = 0.17221$

[4]

iv) Comment:

	Base case	Revised scenario	Change in Probabilities %
Cohort 1	0.458759	0.296496	35.4%
Cohort 2	0.28101	0.17221	38.7%
	61.3%	58.1%	

Cohort 1 - Smoker - regularly eats from outside, no exercise - Heavy drinking, Night shift

Cohort 2 - Non-smoker, irregular on diet & exercise, but non-alcoholic, sleeps 8 hours

- In both cases, with the additional information, the model helped producing lower probability under the revised scenario.
- The reduction seems intuitive as it is incorporating lower age band in the calculation, for not getting discharged within two days.
- The reduction in probability for cohort 2 is higher (38.7%) compare to cohort 1 which reflect the underlying hazard rate for the group
- The gap in the probabilities between the two cohorts narrowing down (61.3% to 58.1%) makes sense with the improvement in the probability allowed.

[2]

[14 Marks]

Solution.20

i) Inputs given:

Month	Failure of feeder ducts
6	1 duct in Kiln 1
11	2 ducts in Kiln 2
14	1 duct in Silo 1 due to major accident in the factory
15	2 ducts in Silo 2
17	1 duct in Limestone work yard
20	1 duct in Kiln 3 wrong handling
22	1 duct in the package yard

j	t_j	d_j	n_j	$\hat{\lambda}_j = d_j/n_j$	$1 - \hat{\lambda}_j$	$\prod_{k=1}^5 (1 - \hat{\lambda}_k)$	Cumulative $\hat{\lambda}_j$
1	6	1	20	0.0500	0.950	0.950	0.0500
2	11	2	19	0.1053	0.895	0.850	0.1553
3	15	2	16	0.1250	0.875	0.744	0.2803
4	17	1	14	0.0714	0.929	0.691	0.3517
5	22	1	12	0.0833	0.917	0.633	0.4350

The estimate of life-time function is,

$$\hat{S}(t) = \begin{cases} 1, & 0 < t < 6 \\ 0.950, & 6 \leq t < 11 \\ 0.850, & 11 \leq t < 15 \\ 0.744, & 15 \leq t < 17 \\ 0.691, & 17 \leq t < 24 \\ 0.633, & 22 \leq t < 24 \end{cases}$$

The Kaplan-Meier estimate of the distribution function is,

$$\hat{F}(t) = 1 - \hat{S}(t) = \begin{cases} 0, & 0 < t < 6 \\ 0.05, & 6 \leq t < 11 \\ 0.15, & 11 \leq t < 15 \\ 0.26, & 15 \leq t < 17 \\ 0.31, & 17 \leq t < 24 \\ 0.37, & 22 \leq t < 24 \end{cases}$$

[5]

ii) Calculation of Nelson-Aalen estimate of life-time distribution

$$\hat{\Lambda}(t) = \sum_{t_i < t} d_j / n_j = \begin{cases} 0, & 0 < t < 6 \\ 0.050, & 6 \leq t < 11 \\ 0.155, & 11 \leq t < 15 \\ 0.280, & 15 \leq t < 17 \\ 0.352, & 17 \leq t < 24 \\ 0.435, & 22 \leq t < 24 \end{cases}$$

$$\hat{S}(t) = e^{-\hat{\Lambda}(t)} = \begin{cases} 1, & 0 < t < 6 \\ e^{-0.050} = 0.951, & 6 \leq t < 11 \\ e^{-0.155} = 0.856, & 11 \leq t < 15 \\ e^{-0.280} = 0.756, & 15 \leq t < 17 \\ e^{-0.352} = 0.703, & 17 \leq t < 24 \\ e^{-0.435} = 0.647, & 22 \leq t < 24 \end{cases}$$

[3]

[8 Marks]

Solution.21

i) Calculation of retention level of surplus treaty which help cede the risks to reinsurer similar to quota share arrangement.

Considering the proportion of business written, total sum assured in-force at the end of 2025 is,

Policy in-force allows for 5% yearly lapse rate

Total sum at risk now = $(16 \times 40\% + 32 \times 35\% + 64 \times 25\%) \times 100 \times (1 - 0.05)^5 = 2,600$

The reinsured sum assured using 50:50 quota share is, $= 50\% \times 2,600 = 1,300$

SA band	Risks reinsured	Mean of Pareto	Expected claim
16	$495 / 2 = 247.6$	0.10	24.8
32	$867 / 2 = 433.3$	0.17	72.2
64	$1,238 / 2 = 619.0$	0.25	154.8
		Total	251.7

Expected claim of $pareto(\alpha, \lambda)$ distribution is $= \lambda/(\alpha-1) * \text{risk reinsured}$, calculate for each SA band

Total expected claim for the risk reinsured $= 50\% * \text{Total risk in-force} * \text{Mean of Pareto}$

[3]

- ii) Working out the retention limit (M1) on the surplus arrangement which help have the same level of reinsured amount considering in-force risk at present, initially it is assumed that the retention limit is less than 16,

$$\Rightarrow 100 * (1 - 0.05)^5 * [(16 - M1) * 40\% + (32 - M1) * 35\% + (64 - M1) * 25\%] = 1,300$$

$$\Rightarrow 33.6 - M1 = 1,300 / (100 * (1 - 0.05)^5)$$

$$\Rightarrow M1 = 33.6 - 16.8 = 16.8$$

→ as the retention limit is higher than the SA covered in the first band, the correct calculation is,

$$\Rightarrow 100 * (1 - 0.05)^5 * [0 * 40\% + (32 - M1) * 35\% + (64 - M1) * 25\%] = 1,300$$

$$\Rightarrow 77.4 * [27.2 - M1 * 60\%] = 1300$$

$$\Rightarrow M1 = (27.2 - 1300/77.4)/60\%$$

$$\Rightarrow M1 = 17.33$$

[3]

- iii) Total risk in-force with insurer at the end of five years after allowing of the decrements,

	Year	2021	2022	2023	2024	2025
	Policies written	25	30	40	55	60
	Policies in-force	25	53	87	134	180
Business mix	Total sum at risk in-force from Product B					
30%	16 SA	120	252	419	641	865
35%	32 SA	280	588	977	1,495	2,018
35%	64 SA	560	1,176	1,954	2,991	4,036
	Total risk in-force	960	2,016	3,350	5,127	6,919

Policy in-force allows for 10% yearly lapse rate

Total sum at risk in-force from Product B at the end of 2025 year = $(16 * 30\% + 32 * 35\% + 64 * 35\%) * 180 = 6,919$

Calculation of retention limit under surplus arrangement for Product B:

Total in-force sum at risk written from Product A = 2,600

Total in-force sum at risk written from Product B = 6,919

So, Total in-force sum at risk written from both = 9,519

Insurer's target over all retention ratio is = 60%

Hence the expected sum at risk to be reinsured under Product B current year,

$$= (1 - 60\%) * 9519 - 1300 = 2,507.5$$

Let us assume the size of retention limit to be calculated is M2 and the calculation of its value is,

$$\Rightarrow 180 * [(16 - M2) * 30\% + (32 - M2) * 35\% + (64 - M2) * 35\%] = 2507.5$$

$$\Rightarrow 38.4 - M2 = 2507.5 / 180$$

$\Rightarrow M2 = 24.48$, as the retention limit is again higher than the first SA band, it needs to be reworked again,

$$\Rightarrow 180 * [0 * 30\% + (32 - M2) * 35\% + (64 - M2) * 35\%] = 2507.5$$

$$\Rightarrow 33.6 - M2 * 70\% = 2507.5 / 180$$

$$\Rightarrow M2 = (33.6 - 2507.5 / 180) / 70\%$$

$$\Rightarrow M2 = 28.12$$

[5]

iv) The final retention limit applied on the product B is 10.

The expected value of the portion of the claim in excess of the limit M (given $X > M$) is: $E[X - M / X > M] = E[X / X > M] - M$

By definition of conditional expectation: $E[X / X > M] = \frac{\int_M^\infty x \cdot f(x) dx}{P(X > M)}$

Use standard normal variable to transform to solve the integral, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

Let $Z = \frac{(\ln(x) - \mu)}{\sigma}$, which implies $X = e^{(\mu + \sigma * Z)}$ and $dx = \sigma e^{(\mu + \sigma * Z)} dz$

and when $x \rightarrow M$, $Z \rightarrow Z_M = \frac{\ln(M) - \mu}{\sigma}$.

$$\int_M^\infty x \cdot f(x) dx \rightarrow \int_{Z_M}^\infty e^{\mu + \sigma * z} * \phi(z) dz \rightarrow \int_{Z_M}^\infty e^{\mu + \sigma * z - z^2/2} dz$$

The exponent $\mu + \sigma * z - z^2/2$ can be rewritten by completing the square: $\mu + \sigma^2/2 - (z - \sigma)^2/2$

The integral now looks like this: $= e^{(\mu + \sigma^2/2)} \left[\frac{1}{\sqrt{2\pi}} \int_{Z_M}^\infty e^{-(z - \sigma)^2/2} dz \right]$

The term inside the brackets is the area under a normal curve shifted by sigma. By substituting $u = z - \sigma$, the limits change from Z_M to $Z_M - \sigma$.

The above integral evaluates to: $1 - \Phi(Z_M - \sigma) = \Phi(\sigma - Z_M)$

$$P(X > M) = 1 - P(X \leq M) = 1 - \Phi(Z_M)$$

Substitute the partial expectation back into the conditional expectation formula:

$$E(X - M / X > M) = e^{(\mu + \sigma^2/2)} * \Phi(\sigma - Z_M) / (1 - \Phi(Z_M)) - M$$

Here, $M=10$

$$\mu = 2.6$$

$$\sigma = 0.8$$

$$Z_M = (\ln(10) - 2.6) / 0.8 = -0.372$$

$$\Phi(Z_M) = 0.355$$

$$\Phi(\sigma - Z_M) = 0.879$$

$$\begin{aligned} E(X - M / X > M) &= e^{\left(2.6 + \frac{0.8^2}{2}\right)} * \Phi(0.8 - (-0.372)) / (1 - \Phi(-0.372)) - 10 \\ &= e^{\left(2.6 + \frac{0.8^2}{2}\right)} * \frac{\Phi(0.8 + 0.372)}{1 - \Phi(-0.372)} - 10 \\ &= 18.5 * 0.879 / (1 - 0.355) - 10 \\ &= 25.3 - 10 \\ &= 15.3, \text{ is the retention limit to be applied for Product A} \end{aligned}$$

[4]
[15 Marks]
