INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

February 2025

CS2 - Risk Modelling and Survival Analysis (Paper A)

Time allowed: 3 Hours 15 Minutes

Total Marks: 100

INDICATIVE SOLUTION

Sol.1 A) False. The correct expression for $p_{AA}(s,t)$ is $e^{-H(t)}$, where $H(t) = \int_s^t \mu(u)du$, not $H(t-s)$.
B) True. If $\mu(t)$ is constant ($\mu(t) = c$), the cumulative hazard simplifies to $H(t) = ct_r$ and $p_{AA}(s,t) = e^{-c(t-s)}$.
C) False. The cumulative hazard $H(t)$ reflects the total accumulated risk over time, not the instantaneous probability of transition.
D) True. Since $\mu(t) \geq 0$, the hazard always adds to the cumulative risk, making $p_{AA}(s,t)$ decrease as t increases.
E) False. $p_{AA}(s,t)$ depends on the exponential of the cumulative hazard, so it remains exponential regardless of whether $\mu(t)$ increases or decreases. A **[3]**

Sol.2 The probability distributions for Xt vary with t, implying time-dependent behavior.
Since the distribution depends on time, the process does not meet the criteria for stationarity.
Answer: (C) The process is neither strictly stationary nor weakly stationary. C **[2]**

Sol.3 a) Correct. The autocovariance depends only on $h = |t-s|$, satisfying weak stationarity.
b) Incorrect. The mean is constant ($\mathbb{E}[X_t] = 0$).
c) Incorrect. Weak stationarity is guaranteed regardless of whether σ^2 changes, as long as σ^2 is defined for all t.
d) Incorrect. The variance $\gamma(0)$ does not grow with t; it is constant.
e) Incorrect. Weak stationarity holds regardless of the sign of a, as long as $|a| < 1$.

Option C (Statement a alone is true) C **[2]**

Sol.4 Insurance companies need to verify if their mortality experience aligns with past data or published life tables to ensure that their premium rates are appropriate for both profitability and competitiveness. Additionally, it allows the insurer to confirm that the observed mortality aligns with pricing assumptions.

- Option B: Both III and V are correct, as these are the main reasons for checking mortality consistency. B **[2]**

Sol.5 **Total Defects Reported:** 5+8+4+6=23
At Risk Each Quarter:

- **Q1:** 500−25=475 500 - 25 = 475500−25=475
- **Q2:** 475−15=460 475 - 15 = 460475−15=460
- **Q3:** 460 (no additional unreachables in Q3)
- **Q4:** 460 (end of tracking period)

Nelson-Aalen Cumulative Hazard Calculation H(t):

$$H(t) = \frac{5}{475} + \frac{8}{460} + \frac{4}{460} + \frac{6}{460} = 0.0105 + 0.0174 + 0.0087 + 0.0130 =$$

Survival Probability S(t):
$S(t) = e^{-H(t)} = e^{-0.0496} \approx 0.9517$ B **[3]**

Sol.6 1. Irreducibility:
   - Buildings 1 and 2 can only go to Building 3.

- Building 3 connects to Buildings 1, 2, 4, and 5, making it a central hub.
- Building 4 connects to Buildings 3 and 6, and Building 5 connects to Buildings 3 and 6.
- Building 6 connects back to Buildings 4 and 5.
- From any building, it is possible to reach any other building through a series of moves (primarily by going through Building 3). Therefore, the chain is irreducible.

2. Aperiodicity:
    - Since Buildings 1 and 2 can only transition to Building 3 in one step, they effectively have a period of 1 when considered individually.
    - However, some buildings (like Buildings 4 and 6) only connect with certain buildings at even-numbered steps (e.g., it takes an even number of steps to return to Building 6 if it moves to Building 4 or 5). This periodic movement suggests that some states may exhibit periodic behavior with period 2.
    - Because of this, the chain is not aperiodic; certain states can only be revisited in even-numbered steps.

Final Answer:

(D) The Markov chain is irreducible but not aperiodic because some states only have access at even-numbered steps. D **[3]**

Sol.7

1. When N=0:If no claims occur, the total claim amount is 0. Thus:

$P(S \leq 3|N=0) = P(S=0) = 1$

$P(S \leq 3 \text{ for } N=0) = P(N=0) \times 1 = 0.4$

2. When N=1:
   The total claim amount can be 1, 2, or 3 (since there is only one claim), and each value has probabilities 0.5, 0.3, and 0.2, respectively. So:

$P(S \leq 3|N=1) = P(S=1) + P(S=2) + P(S=3) = 0.5 + 0.3 + 0.2 =$

$P(S \leq 3 \text{ for } N=1) = P(N=1) \times 1 = 0.3$

3. When N=2:
   Now, we need to calculate the probability that two independent claims sum to 3 or less. The possible combinations for the claims are:

- (1,1) with probability 0.5×0.5=0.25
- (1,2) or (2,1) with probability 0.5×0.3+0.3×0.5=0.30

$P(S < 3|N=2) = 0.25 + 0.3 = 0.55$

$P(S \leq 3 \text{ for } N=2) = P(N=2) \times 0.55 = 0.2 \times 0.55 = 0.11$

4. When N=3:
   Now, we need the probability that the sum of three independent claims is less than or equal to 3. The only way this can happen is if all three claims are equal to 1. The probability of this occurring is:

$P(X_1 = 1, X_2 = 1, X_3 = 1) = 0.5 \times 0.5 \times 0.5 = 0.125$

$P(S \leq 3 \text{ for } N=3) = P(N=3) \times 0.125 = 0.1 \times 0.125 = 0.0125$

$P(S \leq 3) = P(S \leq 3 \mid N=0) \cdot P(N=0) + P(S \leq 3 \mid N=1) \cdot P(N=1)$

$+P(S \leq 3 \mid N=2) \cdot P(N=2) + P(S \leq 3 \mid N=3) \cdot P(N=3)$

$P(S \leq 3) = 0.4 + 0.3 + 0.11 + 0.0125 = 0.8225$ B **[3]**

Sol.8 $h(t \mid Z) = h_0(t)e^{\beta Z}, S_0(t) = e^{-0.02t}$

Covariate $Z = 2$, and $\beta = 0.4$.
Solution:

The survival function under proportional hazards is: $S(t \mid Z) = S_0(t)^{e^{\beta Z}}$

Median survival satisfies $S(t \mid Z) = 0.5$:

$$S(\, t \mid Z\,) = e^{-0.02t - e^{0.8}} = 0.5$$

Solve for t :

$$-0.02t \cdot e^{0.8} = \ln(0.5)$$
$$t = \frac{\ln(0.5)}{-0.02 \cdot e^{0.8}}$$
$$\approx 15.57$$

B **[2]**

Sol.9 The process has the form:

$$\Phi(L) = I - A_1 L - A_2 L^2$$

where:

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -0.3 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}$$

The determinant of $I - A_1 L - A_2 L^2$ leads to the following characteristic equation:

$$\det(I - A_1 L - A_2 L^2) = L^2 - 2.2L + 0.96 - 0$$

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},.$$
$$L = \frac{-(-2.2) \pm \sqrt{(-2.2)^2 - 4(1)(0.96)}}{2(1)} = \frac{2.2 \pm \sqrt{4.84 - 3.84}}{2}.$$
$$L - \frac{2.2 \pm \sqrt{1.0}}{2}.$$
$$L = \frac{2.2 \pm 1}{2} \longrightarrow L = 1.6 \text{ and } L = 0.6.$$

The root $L = 1.6$ lies outside the unit circle, so the system is unstable.

Final Answer:
C) Unstable

C **[2]**

Sol.10

- The PDF of the Gamma(2, 1) distribution is given by:

$$f_{\text{Gamma}}(x) = \frac{1}{\Gamma(2)} x e^{-x} = x e^{-x}, x > 0$$

For large x, the exponential term e^{-x} dominates and decays rapidly to 0.

- The PDF of the Pareto(3, 1) distribution is:

$$f_{\text{Pareto}}(x) = 3x^{-4}, \quad x > 1$$

For large x, the tail behaves like x^{-4}, which decays slower than the exponential term in the Gamma distribution.
Thus, as x⟶∞, the limiting density ratio is:

$$\lim_{x\to\infty}\frac{f_{\text{Gamma}}(x)}{f_{\text{Pareto}}(x)}=\lim_{x\to\infty}\frac{xe^{-x}}{3x^{-4}}=\lim_{x\to\infty}3x^5e^{-x}=0$$

This shows that the Gamma distribution has a lighter tail than the Pareto distribution.
Correct Answer: B (The limiting density ratio is 0, meaning the Gamma distribution has a lighter tail) B **[2]**

Sol.11 We will use the **reduction factor formula** to calculate the projected mortality rate:

$$R_{x,n}=\alpha_x+(1-\alpha_x)(1-f_{x,n})^t$$

Given:

- m65,0=0.008 (current mortality rate at age 65)
- α65=0.0016 (minimum possible mortality rate as a proportion of the current rate)
- f65,15=0.5 (50% of the maximum possible reduction will have occurred by 15 years)
- The time t=30/15=2 (since we are projecting over 30 years, and 50% reduction occurs in 15 years)

Step 1: Apply the Reduction Factor Formula

First, calculate the **reduction factor**:

$$R_{65,30}=\alpha_{65}+(1-\alpha_{65})\left(1-f_{65,15}\right)^2$$

Substitute the given values into the equation:

$$R_{65,30}=0.0016+(1-0.0016)(1-0.5)^2$$
$$=0.0016+0.9984\times(0.5)^2$$
$$=0.0016+0.9984\times0.25$$
$$=0.0016+0.2496=0.2512$$

Step 2: Calculate the Projected Mortality Rate

Now, calculate the projected mortality rate at age 65 in 30 years by multiplying the current mortality rate by the reduction factor:

$$m_{65,30}=m_{65,0}\times R_{65,30}$$

Substitute the values:

m_6530=0.008×0.2512=0.0020096 A **[3]**

Sol.12 Let X(t) represent the number of passengers arriving at the elevator boarding station by time t. X(t) follows a **Poisson process** with parameter λt. The probability that the elevator does not move in the first 90 minutes is the probability that fewer than 4 passengers arrive in the first 90 minutes.

We are looking for P(X(90)≤3), where X(90)∼Poisson(λt)=Poisson(9), since λ=1/10 and t=90.

$$P(X(90)\leq3)=P(X(90)=0)+P(X(90)=1)+P(X(90)=2)+P(X($$

Using the Poisson probability mass function, we compute:

- $P(X(90)=0)=\frac{e^{-9}(9)^{\circ}}{0!}=e^{-9}$
- $P(X(90)=1)=\frac{e^{-2}(9)^1}{1!}=9e^{-9}$
- $P(X(90)=2)=\frac{e^{-2}(9)^2}{2!}=\frac{81e^{-9}}{2}$
- $P(X(90)=3)=\frac{e^{-9}(9)^3}{3!}=\frac{729e^{-9}}{6}$

Adding these together:

$$P(X(90) \leq 3) = e^{-9}\left(1 + 9 + \frac{81}{2} + \frac{729}{6}\right)$$

$$= e^{-9} \times (1 + 9 + 40.5 + 121.5) = e^{-9} \times 172$$

=0.0212

B **[3]**

Sol.13 **Step 1: Calculate Expected Deaths**

The expected number of deaths is calculated as

Ei=Central Exposed to Risk×Mortality Rate.

For age 60-61:

$$E_1 = 47{,}000 \times 0.0080 = 376$$

For age 62-63:

$$E_2 = 43{,}000 \times 0.0095 = 408.5$$

For age 64-65:

$$E_3 = 40{,}000 \times 0.0110 = 440$$

Step 2: Calculate Chi-Squared Statistic

Now, calculate the contribution to the chi-squared statistic for each age group:

For age 60-61:

$$\frac{(370 - 376)^2}{376} = \frac{(-6)^2}{376} = \frac{36}{376} \approx 0.0957$$

For age 62-63:

$$\frac{(380 - 408.5)^2}{408.5} = \frac{(-28.5)^2}{408.5} = \frac{812.25}{408.5} \approx 1.988$$

For age 64-65:

$$\frac{(390 - 440)^2}{440} = \frac{(-50)^2}{440} = \frac{2500}{440} \approx 5.682$$

Step 3: Sum of Chi-Squared Contributions

Summing the individual chi-squared values gives:

$\chi^2 = 0.0957 + 1.988 + 5.682 \approx 7.77$ E **[2]**

Sol.14 Convert to uniform marginals:

$$u = F_X(1) = 1 - e^{-1} \approx 0.632, v = F_Y(0.5) = 1 - e^{-1} \approx 0.632$$

$$C(u, v) = (u^{-2} + v^{-2} - 1)^{-1/2}$$

$$u^{-2} = v^{-2} = (0.632)^{-2} \approx 2.506$$

$$C(u, v) = (2.506 + 2.506 - 1)^{-1/2} = (4.012)^{-1/2}$$

Compute the result:

$$C(u,v) - 1/\sqrt{4.012} \approx 0.5$$

D [3]

Sol.15

1. **Constant Term:** The constant term 2.5 does not contribute to the variance of X_t, as it is a fixed value.
2. **Noise Terms:** $0.4\epsilon_t$, $0.3\epsilon_{t-1}$, $0.2\epsilon_{t-2}$, and $0.1\epsilon_{t-3}$, each scaled by a constant.

$$\text{Var}(X_t) = (0.4^2 + 0.3^2 + 0.2^2 + 0.1^2) \cdot \sigma^2$$

$= 0.30\sigma^2$

D [2]

Sol.16 Step 1: Calculate Precision

$$\text{Precision } = \frac{\text{TP}}{\text{TP} + \text{FP}} = \frac{120}{120+10} = \frac{120}{130} = 0.9231$$

Step 2: Calculate Recall

$$\text{Recall } = \frac{\text{TP}}{\text{TP} + \text{FN}} = \frac{120}{120+30} = \frac{120}{150} = 0.8$$

Step 3: Calculate F1 Score

$$\text{F1 Score } = 2 \times \frac{\text{Precision } \times \text{ Recall}}{\text{Precision } + \text{ Recall}} = 2 \times \frac{0.9231 \times 0.8}{0.9231 + 0.8}$$
$$= 2 \times \frac{0.7385}{1.7231} = 2 \times 0.4286 = 0.856$$

Step 4: Convert F1 Score to Percentage

$$\text{F1 Score } = 0.856 \times 100 = 85.6\%$$

B [2]

Sol.17

1. Markov Property:
   - The process $T(t)$ satisfies the Markov property because its future state depends only on its current state and time t_t not on its past states.
   - Conclusion: Statement a is correct.
2. Stationary Increments:
   - For stationary increments, $T(t+h) - T(t)$ must depend only on h, not t.
   - This happens only if $\mu(T(t),t)$ and $\sigma(T(t),t)$ do not depend on $T(t)$ (i.e., are functions of t alone).
   - Conclusion: Statement b is correct.
3. Geometric Brownian Motion:
   - If $\mu(T(t),t) = \alpha T(t)$ and $\sigma(T(t),t) = \beta T(t)$, the process becomes a geometric Brownian motion.
   - Conclusion: Statement c is correct.
4. Standard Brownian Motion:

- To be a standard Brownian motion, $\mu(T(t), t) = 0$ and $\sigma(T(t), t) = 1$.
- This is a specific case, so it's not always true for $T(t)$.
- Conclusion: Statement d is incorrect.

5. Dependence on $T(t)$:
   1. If $\mu(T(t), t)$ or $\sigma(T(t), t)$ depends on $T(t)$, increments cannot be stationary because the process depends on the current state.
   2. Conclusion: Statement e is correct.

Finally, Option B is correct.

B **[3]**

Sol.18 1. Likelihood for censored data:

The likelihood for censored observations is:

$$L(\lambda) = \prod_{i=1}^{n} S(t_i) = \prod_{i=1}^{n} e^{-\lambda t_i}$$

2. Compute $L(\lambda)$:

$$L(\lambda) = e^{-\lambda(t_1+t_2+t_3)}$$
$$L(\lambda) = e^{-0.05(10+12+15)} - e^{-0.05(\cdot 37)} = e^{-1.85}$$
$$\approx 0.156$$

B **[2]**

Sol.19 1. Solve $\det(\Sigma - \lambda \mathbf{I}) - 0$:

$$\det\begin{bmatrix} 4-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = (4-\lambda)(5-\lambda) - 9 - \lambda^2 - 9\lambda + 11 - 0$$

2. Eigenvalues: $\lambda_1 = 7.541, \lambda_2 = 1.459$.

D **[2]**

Sol.20 Calculate E[S|λ] and Var(S|λ): Given that claims are Poisson distributed and claim amounts are exponential, we first compute these conditional expectations and variances.
Apply the law of total variance: Since the parameter λ varies by vehicle type, we take the expectation and variance over λ to account for the different vehicle categories.

A **[2]**

Sol.21 Since the insider has access to a larger filtration Gt compared to the market's natural filtration Ft (where Ft⊂Gt for all t), the insider possesses more information than the regular market participants. This additional information in Gt allows the insider potentially to predict or anticipate movements in the stock price that would not be predictable under the natural market filtration Ft .
For a stock price modeled as a geometric Brownian motion under Ft, the larger filtration Gt could reveal additional insights, allowing the insider to exploit this information asymmetry and thus gain an advantage in predicting future price movements.

B **[2]**

Sol.22 The formula for ${}_tp_x$ is:

$$ {}_tp_x = \exp\left(-\frac{B}{C}\cdot\left(e^{C(x+t)} - e^{Cx}\right)\right) $$

Substituting the values gives 0.8377 C **[2]**

Sol.23 Directly from Healthy to Dead (HD):

$$ P_{HD} = \frac{0.014}{0.026} \times (1 - e^{-0.026}) = 0.01382 $$

2. From Healthy to Disease Y , then from Disease Y to Dead (HYD):

$$ P_{HYD} = \frac{0.005 \times 0.4}{0.026 \times 0.4} \times (1 - e^{-0.4}) = 0.000871 $$

3. From Healthy to Disease Z, then from Disease Z to Dead (HZD):

$$ P_{HZD} = \frac{0.007 \times 0.7}{0.026 \times 0.7} \times (1 - e^{-0.7}) = 0.001948 $$

Thus, the total probability of dying is:

$$ P_{\text{total}} = 0.01382 + 0.000871 + 0.001948 = 0.016639 $$ B **[4]**

Sol.24 Estimate μ :

$$ \mu = \bar{x} = 24.20 $$

2. Estimate α_1 : Using the formula:

$$ \alpha_1 = \frac{\sum_{i=1}^{599} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{600} (x_i - \bar{x})^2} $$

Substitute the given values:

$$ \alpha_1 = \frac{3{,}122.50}{4{,}256.20} \approx 0.733 $$

3. Estimate σ^2 : Using the formula for the white noise variance:

$$ \sigma^2 = \frac{\sum_{i=1}^{600} (x_i - \bar{x})^2}{600} \times (1 - \alpha_1^2) $$

Substitute $\alpha_1 = 0.733$:

$$ \sigma^2 = \frac{4{,}256.20}{600} \times (1 - 0.733^2) $$
$$ = 7.0937 \times (1 - 0.5373) \approx 7.0937 \times 0.4627 \approx 1.964 $$ C **[3]**

Sol.25 Step 1: Calculate the Mean
Mean for Regular Members $= 1{,}500 \times 40{,}000 \times 0.010 = 600{,}000$
Mean for Special Members $= 300 \times 25{,}000 \times 0.015 - 112{,}500$
Total Mean $= 600{,}000 + 112{,}500 = 712{,}500$
Step 2: Calculate the Variance
Variance for Regular Members $= 1{,}500 \times 40,000^2 \times 0.010 \times (1 - 0.010) = 23{,}760{,}000{,}000$
Variance for Special Members $= 300 \times 25,000^2 \times 0.015 \times (1 - 0.015) = 6{,}646{,}875{,}000$
Total Variance $= 23{,}760{,}000{,}000 + 6{,}646{,}875{,}000 = 30{,}406{,}875{,}000$
Total Variance (in thousands) $= 30{,}406{,}875$

Step 3: Calculate the Skewness
Skewness for Regular Members $= 1{,}500 \times 40,000^3 \times 0.010 \times (1 - 0.010) \times (1 - 2 \times 0.010) - 9.4 \times 10^{12}$
Skewness for Special Members $= 300 \times 25,000^3 \times 0.015 \times (1 - 0.015) \times (1 - 2 \times 0.015) - 1.7 \times 10^{12}$

$$\text{Total Skewness } = 9.4 \times 10^{12} + 1.7 \times 10^{12} = 11.1 \times 10^{12}$$

Coefficient of Skewness:

$$\text{Coefficient of Skewness } = \frac{11.1 \times 10^{12}}{(30.406.875)^{3/2}} = 0.285$$

B **[4]**

Sol.26 Data Limitation: The organization only has 10 years of historical mortality data for ages 60 to 90. This limited period may not be sufficient to accurately capture cohort effects, which require longer data histories to identify patterns unique to specific birth cohorts.
Model Suitability:

- An age-period model would focus on calendar time and use the historical data to model general improvements over time.
- An age-cohort model would attempt to incorporate cohort-specific effects, but with only 10 years of data, it may struggle to accurately assess and separate these effects.

Conclusion: Due to the limited data for observing cohort-specific trends, the age-period model (option C) is more appropriate, as it can provide reasonable projections based on the available calendar time data without overreliance on incomplete cohort effects. C **[2]**

Sol.27 To find the expected lifetime of a new boiler before it is scrapped, we calculate the expected time to breakdown in each phase until it reaches 5 breakdowns.

1. **Expected Time in Each Phase**:
    - For the **first breakdown** (λ = 1/6): Expected time = 1/λ=6 years.
    - For the **second breakdown** (λ = 1/4): Expected time = 1/λ=4 years.
    - For the **third to fifth breakdowns** (λ = 1/2): Each breakdown has an expected time of 1/λ=2 years.
2. **Total Expected Lifetime**:
    - Total time = 6+4+2+2+2=16 years. C **[3]**

Sol.28 Identify the Expression in Terms of Differences:

- We start with the expression $X_{t+3} - 4X_{t+1} + X_{t-2}$.

Rewrite Using Second-Order Differences:

- Recall that the second-order difference $\Delta^2 X_t - \Delta(X_{t+1} - X_t) = X_{t+2} - 2X_{t+1} + X_t$.
- We aim to express $X_{t+3} - 4X_{t+1} + X_{t-2}$ in terms of $\Delta^2 X_t$ and ΔX_t.

Calculate the Equivalent Form:

- After algebraic manipulation, we find that $X_{t+3} - 4X_{t+1} + X_{t-2} - 2\Delta^2 X_t + 3\Delta X_t$. D **[2]**

Sol.29 **Check for Independent Increments**:

- The increments It are independent of past values of Xt−1,Xt−2,… because It is defined independently as a random variable with a fixed probability distribution.
- Therefore, **the process has independent increments**.

Check for the Markov Property:

- Since the future value Xt depends only on the current value Xt−1 and the random increment It, the process satisfies the Markov property.
- The process does not depend on any values before Xt−1, fulfilling the definition of a Markov process.

A **[2]**

Sol.30 Identify Parameters:

- Given $\alpha = 0.5$ and $\beta = 0.3$, calculate $\alpha + \beta = 0.8$ and $\alpha\beta = 0.15$.

Write Down the Yule-Walker Equations:

- For the AR(2) process, the Yule-Walker equations are:

$$\rho_1 - (\alpha + \beta) + \alpha\beta\rho_1 = 0$$
$$\rho_2 - (\alpha + \beta)\rho_1 + \alpha\beta = 0$$

- Substitute $\alpha + \beta = 0.8$ and $\alpha\beta = 0.15$ into these equations.

Substitute and Simplify:

- First equation: $\rho_1 - 0.8 + 0.15\rho_1 = 0$.

Second equation: $\rho_2 - 0.8\rho_1 + 0.15 - 0$. A **[2]**

Sol.31 Insurer's Payment for One Claim:

$$\mathbb{E}[Y] = 0.6 \cdot \int_d^\infty (x - d)\frac{1}{10{,}000}e^{-x/10{,}000}dx$$

After integration:

$$\mathbb{E}[Y] = 0.6 \cdot 10.000 \cdot e^{-0.5} = 3639$$

E **[3]**

Sol.32

Time (months)	Events Observed	Patients at Risk	Survival Probability at Interval	Cumulative S Probability
0 - 3	2 deaths	20	$\frac{20-2}{20} = 0.90$	0.90
3 - 6	1 censored	18	$\frac{18}{18} = 1.00$	0.90×1.00
6 - 9	3 deaths	18	$\frac{18-3}{18} = 0.833$	0.90×0.83
9 - 12	2 deaths	15	$\frac{15-2}{15} = 0.867$	0.75×0.86

B **[3]**

Sol.33 Calculate the probability of **correct ensemble prediction**:

- **Case 1:** All 3 models are correct:
  p1×p2×p3=0.85×0.90×0.80=0.612
- **Case 2:** Exactly 2 models are correct:
  - Model 1 and Model 2 are correct, Model 3 is incorrect:
    p1×p2×(1−p3)=0.85×0.90×0.20=0.153

- Model 1 and Model 3 are correct, Model 2 is incorrect: p1×(1−p2)×p3=0.85×0.10×0.80=0.068
- Model 2 and Model 3 are correct, Model 1 is incorrect: (1−p1)×p2×p3=0.15×0.90×0.80=0.108

Add up all these probabilities:
0.612+0.153+0.068+0.108=0.891≈89.2%

E **[3]**

Sol.34 $Q = n(n+2)\sum_{k=1}^{m} \frac{\rho_k^2}{n-k}$

34. Calculate each term $\frac{\rho_k^2}{n-k}$:

- For $k = 1$: $\frac{(0.05)^2}{300-1} = \frac{0.0025}{299} \approx 8.36 \times 10^{-6}$
- For $k = 2$: $\frac{(-0.10)^2}{300-2} = \frac{0.01}{298} \approx 3.36 \times 10^{-5}$
- For $k = 3$: $\frac{(0.07)^2}{300-3} = \frac{0.0049}{297} \approx 1.65 \times 10^{-5}$
- For $k = 4$: $\frac{(-0.04)^2}{300-4} = \frac{0.0016}{296} \approx 5.41 \times 10^{-6}$
- For $k = 5$: $\frac{(0.03)^2}{300-5} = \frac{0.0009}{295} \approx 3.05 \times 10^{-6}$
- For $k = 6$: $\frac{(-0.02)^2}{300-6} = \frac{0.0004}{294} \approx 1.36 \times 10^{-6}$

2. Sum these terms:

$$\sum_{k=1}^{6} \frac{\rho_k^2}{n-k} \approx 8.36 \times 10^{-6} + 3.36 \times 10^{-5} + 1.65 \times 10^{-5} + 5.41 \times 10^{-6} + 3.05 \times 10^{-6} + 1.36 \times 10^{-6} \approx 6.32 \times 10^{-5}$$

3. Substitute into the Ljung-Box formula:

$$Q = 300 \times 302 \times 6.32 \times 10^{-5} \approx 6.98$$

B **[3]**

Sol.35 The linearity property of covariance states that for any two random variables X and Y and any third variable W,
Cov(X+Y,W)=Cov(X,W)+Cov(Y,W)
This property holds regardless of whether W (or X and Y) is stationary. It is simply a result of the algebraic property of covariance.

B **[2]**

Sol.36 Since both μ and ν are constant rates, the probability of survival (i.e., still working) can be expressed as:

$$p_{WW}(t,x) = e^{-(\mu+\nu)t}$$

A **[2]**

Sol.37 **Stationarity**: Since it is an MA process with a finite number of terms, Xt is **stationary** by definition.
Not White Noise: Although stationary, Xt is not white noise because it has autocorrelation at lags 1 and 2 due to the dependency on et−1 and et−2.

B **[2]**

Sol.38

- Independent deaths: The probability of paying the benefit under the assumption of independent deaths is:

$P(\text{ second death }) = (1 - p_{70}^{10}) \times (1 - p_{70}^{10}) = (0.35) \times (0.35) = 0.1225$

- Clayton copula: Using the Clayton copula formula with $\alpha = 0.5$:

$$C(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-\frac{1}{\alpha}}$$

Substituting $u = v = 0.35$:

$$C(0.35, 0.35) = (0.35^{-0.5} + 0.35^{-0.5} - 1)^{-2} = 0.1448$$

Thus, the increase in probability is:

$$0.1448 - 0.1225 = 0.0223$$

A **[3]**

Sol.39 **Rate Interval**:

- The deaths were observed for individuals aged between **59½ and 60½**. Thus, the rate interval is: [59.5,60.5]

Age for Estimating μ:

- The force of mortality, μ, is typically estimated at the midpoint of the rate interval.
- Therefore, the age at which μ can be estimated is: 60

Formula for Estimating q:

- The probability of death q for this group can be estimated using: q≈1−e^− q≈1−e^−μ59.5

B **[2]**

Sol.40

1. $\beta = (\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^\top \mathbf{y}$
2. Compute $\mathbf{X}^\top \mathbf{X} = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$.
3. Add $\lambda \mathbf{I}$: $\mathbf{X}^\top \mathbf{X} + 5\mathbf{I} - \begin{bmatrix} 40 & 44 \\ 44 & 61 \end{bmatrix}$.
4. Invert the matrix: $(\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I})^{-1}$.
5. Compute $\mathbf{X}^\top \mathbf{y} = \begin{bmatrix} 22 \\ 28 \end{bmatrix}$.
6. Solve $\beta = (\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^\top \mathbf{y}$.

Final Answer: $\beta = \begin{bmatrix} 0.218 \\ 0.302 \end{bmatrix}$.

B **[3]**

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