INSTITUTE OF ACTUARIES OF INDIA

Subject SP6 - Financial Derivatives

May 2024 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) If the continuous-time processes X_t and Y_t are martingales under the same measure, then there is a unique process f_t such that $dY_t=f_t\, dX_t$ and this process is previsible. (1)

ii) To prove that X_t is a martingale we need to show that $E[X_t|Fs] = X_s$, for $t \geq s$.

$$E[V_t^2] = t = s + t - s \; for \; t > s$$
$$Thus$$
$$E[V_t^2 W_t|F_s] = E[V_t^2|F_s] \times E[W_t|F_s] = (V_s^2 + t - s)W_s$$

$$\mathrm{E}\left[\int_0^t \mathrm{W_u du}\,\middle|\mathrm{Fs}\right] = \int_0^s \mathrm{W_u du} + \mathrm{W_s}(\mathrm{t}-\mathrm{s})$$
$$\mathrm{E}[\mathrm{X_t}|\mathrm{Fs}] = (\mathrm{V_s^2} + \mathrm{t} - \mathrm{s})\mathrm{W_s} - \int_0^s \mathrm{W_u du} + \mathrm{W_s}(\mathrm{t}-\mathrm{s})$$
$$\mathrm{E}[\mathrm{X_t}|\mathrm{Fs}] = \mathrm{V_s^2 W_s} - \int_0^s \mathrm{W_u du} = \mathrm{X_s}$$

(5)

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Solution 2:

i) When interest rates are high, there is low demand for funds from borrowers and so interest rates decline.
However, when interest rates are low the demand for funds on the part of borrowers increases and so interest rates tend to rise.
One of the ways that Central banks pursue inflation targets via monetary policy, and that could lead to interest rates becoming mean reverting.
An excessively high interest rate may dampen economic growth which may cause a cyclical decline of the interest rate.
If interest rates turn negative investors will hold cash instead, reducing the demand for borrowing and also the effectiveness of monetary policy. (2)

ii)

$$P(t,T) = \exp\left(-R(t,T)(T-t)\right)$$

$$R(t,T) = -\frac{1}{T-t}\ln P(t,T)$$

$$R(t,T) = -\frac{\ln A(t,T)}{T-t} + \frac{\ln(\exp(B(t,T)r(t)))}{T-t}$$

$$R(t,T) = \frac{B(t,T)r(t)}{T-t} - \frac{\ln A(t,T)}{T-t}$$

(2)

iii) *R(t, T)* is linearly dependent on *r(t)*. *r(t)* determines the level of the term structure of the continuously compounded spot curve at time t. Thus, the general shape of the term structure at time t is independent of *r(t)*. (2)

iv) In a risk neutral world, all asset prices must have an expected return of the risk free rate. Allowing for mean reversion of bond prices would prevent free stochastic/random movement. This means that the expected return could deviate from the risk free rate, imply arbitrage opportunities, and as bonds are tradable instruments (as opposed to interest rates). (2)

[8]

Solution 3:

i) As per the Merton model Equity of company can be considered as a put option on company's enterprise value with strike price of Debt hence risk in equity would effectively means that company should be able to service the debt better. There are multiple models used for determining the debt of the company including structure model and reduced form model. (3)

ii) The CAPM is a model for pricing an individual security or portfolio. For individual securities, we make use of the security market line (SML) and its relation to expected return and systematic risk (beta) to show how the market must price individual securities in relation to their security risk class. The SML enables us to calculate the reward-to-risk ratio for any security in relation to that of the overall market. Therefore, when the expected rate of return for any security is deflated by its beta coefficient, the reward-to-risk ratio for any individual security in the market is equal to the market reward-to-risk ratio, thus:

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f)$$

(4)

iii) Beta in CAPM model can be considered as a proxy of risk as observed in the equity market. As probability of bond default increases the stock price will become more volatile leading to a higher beta. (3)

iv) Assuming that default happens at the end of the year, the following table calculates the default value and the 6% default is closer to the par value of the bond.

(7)

[17]

Default rate	4%								
Nominal	-1000								
Year Cashflow	Start	Start of year	Default During the year	Recovery amount	End of year capital	Coupon	Redemption	Total(Coupon+ recovery)	Discounting
1		1,000	40	20	960	80		100	0.952381
2	80	960	38	19	922	77		96	0.9070295
3	80	922	37	18	885	74	885	977	0.8638376
								Bond Price	1026.1924

Default rate	6%								
Nominal	- 1000								
Year Cashflow	Start	Start of year	Default During the year	Recovery amount	End of year capital	Coupon	Redemption	Total(Coupon+ recovery)	Discounting
1		1,000	60	30	940	80		110	0.952381
2	0	940	56	28	884	75		103	0.9070295
3	0	884	53	27	831	71	831	928	0.8638376
								Bond Price	1000

Solution 4

i) Default probability in year 4 is 0.97*0.97*0.97*0.03 = 0.02738 or ~2.74%. (2)

ii) Probability of survival at the end of 5 years is 0.97*0.97*0.97*0.97*0.97 = 0.858734
~85.87%.

(2)

iii) Calculation of PV of payments; principal = $1

Time (yrs)	Default probability	Survival Probability	Expected Payment	Discount Factor	PV of Exp Payment
1	0.0300	0.9700	0.9700*s*	0.9231	0.8954*s*
2	0.0291	0.9409	0.9409s	0.8521	0.8018*s*
3	0.0282	0.9127	0.9127s	0.7866	0.7179*s*
4	0.0274	0.8853	0.8853s	0.7261	0.6429*s*
5	0.0266	0.8587	0.8587s	0.6703	0.5756*s*
Total					3.6336s

PV of Accrual Payment Made in Event of a Default. (Principal=$1)

Time	Default Probability	Expected Accrual Payment	Discount Factor	PV of Payment
0.5	0.0300	0.0150*s*	0.9608	0.0144*s*
1.5	0.0291	0.0146s	0.8869	0.0129*s*
2.5	0.0282	0.0141*s*	0.8187	0.0116*s*
3.5	0.0274	0.0137*s*	0.7558	0.0103*s*
4.5	0.0266	0.0133*s*	0.6977	0.0093*s*
Total				0.0585*s*

PV of expected payments is 3.6336s + 0.0585s = 3.6921s

(4)

iv) Present Value of the expected payoff in the event of a default

Time (yrs)	Default Prob.	Rec. Rate	Expected Payoff	Discount Factor	PV of Exp. Payoff
0.5	0.03	0.3	0.0210	0.9608	0.0202
1.5	0.0291	0.3	0.0204	0.8869	0.0181
2.5	0.0282	0.3	0.0198	0.8187	0.0162

3.5	0.0274	0.3	0.0192	0.7558	0.0145
4.5	0.0266	0.3	0.0186	0.6977	0.0130
Total					0.0819

PV of expected payment in the event of default is 0.0819

(2)

v) Breakeven CDS spread: 3.6921s = 0.0819 or s = 0.02218 or 221.8 bps (3)

vi) Value of a CDS negotiated at an earlier point in time with a CDS spread of 323.98as a function of the principal:
(3.6921 * 0.032398) – 0.0819 = 0.0377 times the principal.

102.18 bps benefits is arising from the bond having survived five years.

(4)
[17]

Solution 5:

i) Using Black-Scholes to price the Call option:
$d1 = (\ln(500/600) + (5\% + 0.5*22.5\%*22.5\%))/22.5\% = -0.4756$
$d2 = d1 - 22.5\% = -0.7006$
So, $C = 500*N(-0.4756) - 600*e^{(-0.05*1)} *N(-0.7006) = 20.60$
Similarly, using the Black-Scholes to price the Put option:
$d1 = (\ln(500/400) + (5\% + 0.5*22.5\%*22.5\%))/22.5\% = 1.3265$
$d2 = d1 - 22.5\% = 1.1015$
Thus, $P = 400*e^{(-0.05 * 1)} *N(-1.1015) - 500*N(-1.3265) = 5.33$

An asset that pays Xi in ith scenario is worth $[X1p1 + X2p2 + X3p3 + X4p4]\ e^{(-0.05 *1)}$
Bonds will be priced correctly because:
$[e^{(0.05*1)}p1 + e^{(0.05*1)}p2 + e^{(0.05*1)}p3 + e^{(0.05*1)}p4]\ e^{(-0.05*1)} = 1$
For the Call to be priced properly: $[0p1 + 0p2 + 0p3 + 100p4]\ e^{(-0.05*1)} = 20.60$
So, $p4 = 0.217$
For the Put to be priced properly: $[100p1 + 0p2 + 0p3 + 0p4]\ e^{(-0.05*1)} = 5.33$
So $p1 = 0.056$
Note that p's add up to 1, so $p3 = 1 - 0.056 - 0.217 - p2 = 0.727 - p2$
For equities to be priced properly:
$[300*0.056 + 450p2 + 550(0.727 - p2) + 700*0.217]\ e^{(-0.05*1)} = 500$
Thus, $p2 = 0.428$ and $p3 = 0.299$.

(10)

ii) These answers are in reference to the calculations in part (i).
Deriving the p's involves solving four simultaneous equations as before, but with different values for the Call and the Put.
The Call option has lower volatility so will be lower in price and since the Call goes down in value, $p4$ will be lower.
The Put option has higher volatility so will be higher in price and since the Put has become more expensive, $p1$ will be higher.

To quantify impact on $p2$ and $p3$, we note that:

- $p2 + p3$ is constrained by the four probabilities summing to 1.
- the expected one-period equity price in the risk neutral measure will be unchanged,
  i.e. [300*$p1$ + 450$p2$ + 550$p3$ + 700*$p4$] $e^{(-0.05*1)}$ = 500

Now, if we need to leave the expected value unchanged after $p1$ rises and $p4$ falls, we need to increase weighting on scenario 3 at the expense of scenario 2.
Hence $p2$ will be lower and $p3$ will be higher.

(4)

iii)

- For the low end of the equity prices: The distribution with the volatility skew will have a fatter tail as compared to the distribution without the volatility skew.
- For the high end of the equity prices: The distribution with the volatility skew will have a thinner tail as compared to the distribution without the volatility skew.
- For the mean distribution: The mean of the distribution with the volatility skew will be shifted to the right (towards higher equity prices) as compared to the one without the volatility skew.

(3)

iv) The call option price depends on the stock price and its volatility, and the volatility is also dependent on the stock price.

$$dC(S,\sigma) = \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma$$

$$\frac{dC(S,\sigma)}{dS} = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \sigma} \times \frac{d\sigma}{dS}$$

$$\frac{dC(S,\sigma)}{dS} = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \sigma} \times \frac{d}{dS}(\alpha S + \beta)$$

$$\therefore \frac{dC(S,\sigma)}{dS} = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \sigma}\alpha$$

$$\textit{The adjusted delta is equal to unadjusted delta plus } \alpha * \textit{Vega}$$

(8)

[25]

<u>Solution 6:</u>

i) Real World projections of interest rates will allow CCX to understand how interest rates could likely evolve in different scenarios. For example, CCX might want to understand whether any initial margin will cover extreme interest rate stresses. Further, Risk neutral projections would be helpful for CCX to understand the complete term structure of interest rates at a given point during the real-world projection. Also, CCX might want to calculate arbitrage-free prices of the securities after 10 years of the real-world projection.

(2)

ii) The SDE is dr(t) = [a(b – r(t))]dt + σ dz(t), where

r(t) is the short interest rate under the real world measure
a, b and σ are constants
z(t) is Brownian motion

CCX needs to calibrate the parameters based on historical movements in the short rate. The data can be a daily, monthly or annual frequency. Parameters can then be determined by either a maximum-likelihood approach or using linear regression. This will let CCX determine values of the constants.

(4)

iii) Suppose that the price of the security has a SDE: $df = f[\mu\, dt + s\, dz]$

The market price of risk is:

$$\lambda = \frac{\mu - r}{s}$$

Where μ and s are the expected growth rate and volatility of f respectively and r is the risk free rate.

(2)

iv) Apply Girsanov's theorem – thus, moving to the risk neutral world sets the instantaneous return to $r = \mu - \sigma\lambda$; … but the volatility of r will remain the same.

Upon substitution, we get,

$$dr = [a(b-r) - \sigma\lambda]dt + \sigma dz = [a(b^* - r)]dt + \sigma dz,$$

where $b^* = [b - (\lambda\sigma/a)]$

(3)

v) The Vasicek model only contains one source of randomness and thus is a one factor model. The Hull and White model on the other hand is a two-factor model (HW) as it contains two sources of randomness and thus it provides a richer pattern of both term structure movements and volatilities than one factor models.

The HW model should therefore provide CCX with a greater range of interest rate scenarios to model. For example, in the Vasicek model, all forward rates along the curve are perfectly correlated, which could mask exposures to counterparties that have different interest rate positions at different parts of the curve.

Further, the Vasicek model struggles to fit some short rate term structures as the parameters do not vary with time. HW model, on the other hand, uses a deterministic function for the mean reversion level set to be consistent with the current yield curve and bond prices, so CCX will be able to replicate current prices of instruments relatively easily.

At the same time, it might be relevant to note that replicating current prices may be of secondary importance to CCX, particularly if we are projecting interest rates for many years into the future in a real-world scenario.

(4)

vi) CCX has set the initial margin levels with respect to the notional of the contract, which is sensible given that the market value will likely be zero on execution. The level of initial margin requirements could equate to a 20bps movement in interest rates given that (taking a 5-year swap) 1% across 5 years = 20bps.

After 5 years, the basis point equivalent stress will materially depend on the duration of the instrument. If we take a 10-year instrument as an example, 2% across 10 years = 20bps.

The key function of the initial margin is to provide a buffer for movements over a one-day period before the margin account is marked to market at the end of the day. Daily interest rates rarely move more than 20bps, so this should add a substantial amount of protection for typical day; but it could prove insufficient in the extreme stresses which pose the most material counterparty risk.

Thus, specifying different margin requirements by term is sensible as longer dated swaps will be more sensitive to a given parallel movement in interest rates. However, the breakdown by term is limited to just two “buckets”, which may not sufficiently reflect the different underlying risk by term.

Counterparties can also “game” the margin requirements (for example, by executing a 4.9 year swap with a larger notional rather than a 6 year swap). Determining the initial margin using the outstanding term will mean there will be a sharp reduction in the margin accounts as soon the term on longer dated instruments contracts to below 5 years, which may not be ideal.

Requiring cash to be posted should reduce any second-order counterparty risk from collateral defaulting at the same time as the counterparty.

However, counterparty risk could still be present from the cash is this includes short term money market instruments. The margin requirements are the same for all counterparties which may not incentivize risk management by the counterparties. For example, CCX could require more margin from riskier counterparties with lower credit ratings.

The margin requirements make no allowance for offsetting between payer and receiver swaps, which will mean large amounts of margin is collected where there is limited counterparty exposure.

When setting the initial margin levels, it may be more effective to allow for offsetting between different instruments and possibly different terms to ensure greater initial margin is collected for the largest exposures. The margin requirements are easy to understand for counterparties, so there should be limited risk of insufficient or incorrect margin being posted.

(6)

vii) **Advantages**

Financial system becomes safer because:

- Bilateral credit risk between counterparties is removed.
- Variation margins are paid daily and mark to market losses are covered.
- Even in case of a default, an initial margin can be utilized.
- Possible support from central bank if liquidity crunch happens as it is authorized by the regulator.
- Risk management practices of clearing corporations are generally more robust.
- The presence of a clearing house can lead to OTC markets becoming deeper as the possibility of credit risk is significantly reduced.

Disadvantages

- Participants generally go in for OTC products to get customized products. Clearing houses generally deal with standardized products. This can impact the risk management capability of the clearing house.

- Systemic risk is not reduced but transferred to the clearing house. Consequences of clearing house failure can be catastrophic to the financial system, and this risk will be high during the initial stages when the concept is tested for the first time.

(6)

[27]

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