INSTITUTE OF ACTUARIES OF INDIA

Subject CM2A - Financial Engineering and Loss Reserving

May 2024 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1: III [2]
Solution 2: II [2]
Solution 3: III [2]
Solution 4: III [2]
Solution 5: II [4]
Solution 6: I [2]
Solution 7: III [2]
Solution 8: II [2]
Solution 9: II [2]
Solution 10: IV [2]
Solution 11: II [4]
Solution 12: II [2]
Solution 13: I [2]
Solution 14: II [4]
Solution 15: III [2]
Solution 16: I [2]
Solution 17: IV [2]

Solution 18:

i) Given that

$dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t$

Let $f(S_t) = ln(S_t)$

Applying the Taylor's series formula to the above function, we get

$df(S_t) = d(ln(S_t)) = \frac{1}{S_t} dS_t + \frac{1}{2}(-\frac{1}{S_t^2})(dS_t)^2$

$= \frac{1}{S_t}((\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t) - \frac{1}{S_t^2}((\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t)^2$

$= (\mu + \frac{1}{2}\sigma^2) dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$

$= \mu dt + \sigma dW_t$

Integrating we get

$ln(S_t) - ln(S_0) = \mu t + \sigma W_t$

Therefore

$\boldsymbol{S_t = S_0 exp(\mu t + \sigma W_t)}$

Now $\boldsymbol{D_t = B_t^{-1} S_t = e^{-rt} S_t = S_0 exp((\mu - r)t + \sigma W_t)}$ ---------(A)

Comparing the $\boldsymbol{S_t}$ **and** $\boldsymbol{D_t}$ it is evident that

$\boldsymbol{d D_t = (\mu - r + \frac{1}{2}\sigma^2) D_t dt + \sigma D_t dW_t}$ (7)

Alternate Solution

Given that

$dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t$

and $\boldsymbol{D_t = B_t^{-1} S_t = e^{-rt} S_t}$

We have

$\frac{\partial D_t}{\partial t}$ = -re^{-rt} S_t =-r D_t

$\frac{\partial D_t}{\partial S_t}$ = e^{-rt}

$\frac{\partial^2 D_t}{\partial S_t^2}$ = 0

Applying Ito's Lemma we get

$$d\,D_t = \frac{\partial D_t}{\partial t}\,dt + \frac{\partial D_t}{\partial S_t}dS_t + \frac{\partial^2 D_t}{\partial S_t^2}dS_t^2$$

$$= -rD_t\,dt + e^{-rt}((\mu + \tfrac{1}{2}\sigma^2)\,S_t\,dt + \sigma S_t\,dW_t)$$

$$= -rD_t\,dt + D_t((\mu + \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dW_t)$$

$$= D_t((\mu - r + \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dW_t)$$

ii) Consider a dynamic portfolio (ϕ_t, ψ_t) consisting of ϕ_t units of *St* and ψ_t units of *Bt*.

A portfolio is ***self-financing*** if and only if changes in its value depend only on changes in the prices of the assets constituting the portfolio.

Mathematically, if V_t denotes the value of the portfolio (ϕ_t, ψ_t), then the portfolio is self-financing if and only if
$dV_t = \phi_t\, dSt + \psi_t\, dBt$

where ϕ_t and ψ_t are previsible. (1)

A ***replicating strategy*** for *X* is a strategy which involves investing in previsible quantities (ϕ_t and ψ_t) of stock and risk-free bonds, such that:
the portfolio (ϕ_t, ψ_t) of stocks and bonds will be self-financing
the portfolio (ϕ_t, ψ_t) will have terminal value equal to the magnitude of the claim, i.e.

$V_T = \phi_T S_T + \psi_T B_T = X$,

which means that the portfolio cash flows at claim exercise date match the cash flows under the claim.

(2)

(3)

iii) Steps are as follows.

(1) Apply the Cameron-Martin-Girsanov theorem to D_t .This states that there exists a new probability measure, say *Q*, equivalent to the current measure, such that D_t =e^{-rt} S_t is a *Q*-artingale,

$$d\,D_t = \sigma\, D_t\, d\hat{W}_t$$

where $\hat{W}_t$ is a standard Brownian motion under *Q*.

(2) Define:

$V_t = e^{-r(T-t)} E_Q\, [X/ F_t]$

We propose that this is the value of the claim.

(3) Form the discounted expected claim process E_t under measure Q:

$E_t = E[B_T^{-1}X \mid \mathbf{F}_t] = e^{-rT} E_Q [X/ F_t] = e^{-rt} V_t$

where $\mathbf{F}_t$ is the history of the process up to time t.

This process is a Q-Martingale, which can be demonstrated using the properties of Martingale and the Tower Law of conditional probabilities.

(4) Invoke the Martingale Representation Theorem (MRT) which states that there is a previsible function ϕt such that:
$dE_t = \phi_t \, dD_t$

(5) Consider a replication strategy of holding ϕ_t units of stock, where ϕ_t is chosen based on the MRT, and

Let $\psi_t = E_t - \phi_t D_t$ of risk free bonds.

Firstly, we show that this portfolio replicates the value of the claim.

The value of the portfolio at any time t is V_t. Also $V_n=C_n$. Therefore the hedging strategy is

(ψ_t, ϕ_t) is replicating and so by no arbitrage , V_t is the fair price of derivative at time t.

(9)

[19]

Solution 19:

i) At the end of one year the investment will be worth $1000(x + (1 - x) * 1.5) = 1500 - 500x$ with probability 0.6
or $1000(x + (1 - x) * 0.5) = 500 + 500x$ with probability 0.4
Therefore, the expected utility is $0.6 * \ln(1500 - 500x) + 0.4 * \ln(500 + 500x)$ (3)

ii) Differentiate expected utility: $U'(x) = -500 * \frac{0.6}{1500-500x} + 500 * \frac{0.4}{500+50x}$

Set equal to zero and solve for x:

0.4(1500-500x) =0.6 (500+500x)
=> x= 0.6
Check that this is a maximum:

$U''(x) = -2{,}50{,}000 * \frac{0.6}{(1500-500x)^2} - 2{,}50{,}000 * \frac{0.4}{(500+50x)^2} < 0$

Therefore, the investor should invest 600 in Asset A and 400 in Asset B (5)

[8]

Solution 20:

i)

a) X = (90000/100000)-1 = -10%
P(X<-10%) = P(Z<-3.09)=0.1% (2)

b) P(Z<(t-7%)/5.5%) =0.005

(t-7%)/5.5% =-2.5758

t= -7.1669%

1,00,000 * (1-7.1669%) = 92,833 (3)

ii) P(X≤-7.1%) = P(Z≤-2.56) =0.00518

P(X>7%) = 0.5 (as 7% is the mean)

P(-7.1%<X≤7%) = 1-0.5-0.00518 = 0.49482

Expected payout = 92,000 x 0.00518 + 95,000 x 0.49482+ 1,20,000 x 0.5

= 1,07,484.46 (4)

iii)

a) 0% . Payout is discrete and has 3 payouts all greater than 90,000 and hence shortfall probability is 0 (1)

b) Probability payout is ≤ 92,000 is 0.52% therefore the 99.5% VaR is 92,000 (1)

[11]

Solution 21:

i) The general form for the incremental claims C_{ij} can be written as:

$C_{ij} = r_j \, s_i \, x_{i+j} + e_{ij}$

where:

r_j is the development factor for year j, representing the proportion of claim payments in year j. Each r_j is independent of the origin year i.

s_i is a parameter varying by origin year, i, representing the exposure, for example the number of claims incurred in the origin year i.

x_{i+j} is a parameter varying by calendar year, for example representing inflation.

e_{ij} is an error term. (4)

ii) Development factors are: Year 3 = 1900 / 1800 = 1.05556
Year 2 = (1800+1850) / (1300+1400) = 1.35185
1 – 1/f = 1 – 1 / (1.05556 x1.35185) = 0.29921
Emerging liability for 2023 = 3000 x 0.8 x 0.29921 = 718.1
Reported liability = 1500
Ultimate liability = 1500 +718.1 = 2218.1
Reserve = 2218.1 – 1000 = 1218.1 (6)

[10]

Solution 22: The risk neutral and real world probability measures are connected by the market price of risk, which equals c.

The stock price and derivative price are given by:
$S_t = S_0 \exp(0.2B_t + 0.2t)$

D_t = 2exp $(0.6B_t + 0.39t)$

If we convert these using the CMG theorem to a standard Brownian motion under the risk-neutral measure and assume that

$ß_t = B_t + ct$, they become:

$S_t = S_0 \exp(0.2(ß_t - ct) + 0.2t) = S_0 \exp(0.2(1 - c)t + 0.2\,ß_t)$
$D_t = 2\exp(0.6(ß_t - ct) + 0.39t) = 2\exp((0.39 - 0.6c)t + 0.6\,ß_t)$

These processes are both geometric Brownian motions and the corresponding SDEs are:

$$dS_t = S_t\left((0.2(1-c) + \frac{0.2^2}{2})\,dt + 0.2d\,ß_t\right) = S_t\left(((0.22-0.2c))dt + 0.2d\,ß_t\right)$$

$$dD_t = D_t\left((0.39 - 0.6c + \frac{0.6^2}{2})dt + 0.6d\,ß_t\right) = D_t\left((0.57 - 0.6c)\,dt + 0.6d\,ß_t\right)$$

Under the risk-neutral measure both assets have the same rate of drift (equal to the risk free rate).

This means:
0.22 − 0.2c = 0.57 − 0.6c
0.4c = 0.35
c = 0.875 **[6]**

Solution 23: The market price of risk is:

$$\frac{E_{M-r}}{\sigma_M}$$

Where

- r is the risk-free interest rate
- E_M is the expected return on the market portfolio consisting of all risky assets
- σ_M is the standard deviation of the return on the market portfolio.

Since Asset 1 always gives the same return of 4%, it is risk-free. So the risk-free interest rate is r =4% .
Assets 2 and 3, with the capitalisations shown, constitute the market portfolio of risky assets.
The total capitalisation of the market is
20,000 + 80,000 = 1,00,000

The table below shows the possible returns on this market portfolio:

State	Probability	Return	Return (%)
1	0.2	5% x 20,000 + 6% x 80,000 = 5,800	5.8%
2	0.3	12% x 20,000 + 5% x 80,000 = 6,400	6.4%
3	0.1	2% x 20,000 + 3% x 80,000 = 2,800	2.8%
4	0.4	1% x 20,000 + 7% x 80,000 = 5,800	5.8%

So, the expected return is:
E_M = 0.2 x 5.8% + … + 0.4 x 5.8% = 5.68%

The variance of the return is:

$\sigma^2_M = 0.2 \times (5.8\%)^2 + \ldots + 0.4 \times (5.8\%)^2 - (5.68\%)^2$
$= 0.9936\%\%$
$= (0.9968\%)^2$

So, the market price of risk is:

$\frac{E_{M-r}}{\sigma_M} = \frac{5.68\% - 4\%}{0.9968\%} = 1.6854$

This shows the extra expected return (over and above the risk-free rate) per unit of extra risk taken (as measured by the standard deviation) by investing in risky assets. **[6]**

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