Institute of Actuaries of India

Subject CM1A – Actuarial Mathematics (Paper A)

July 2022 Examination

INDICATIVE SOLUTION

Introduction
The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

As per the first two conditions of Redington's theory of immunization:

- present value of assets and liabilities should be equal
- volatility of assets and liabilities should be equal [1]

i)

Present value of liabilities $[V[i]_L]$ =
$x*1.06^{-1} + 2x*1.06^{-2} + 3.5x*1.06^{-3} + 4.5x*1.06^{-4} + 5x*1.06^{-5} + 100*1.06^{-6}$
$= 12.9628x + 70.4961$ [1.5]

Present value of assets $[V[i]_A]$ =
$10*1.06^{-1} + 10*1.06^{-2} + 20*1.06^{-5} + 30*1.06^{-6} + y*1.06^{-7}$
$= 54.4279 + 0.66505y$ [1]

As per condition 1:
$12.9628x + 70.4961 = 54.4279 + 0.66505y$ [0.5]
$x = 0.051305y – 1.2396$ -------------------------- [A] [0.5]

Volatility of liabilities is $-V'[i]_L / V[i]_L$, where $-V'[i]_L$ is
$x*1.06^{-2} + 2*2x*1.06^{-3} + 3*3.5x*1.06^{-4} + 4*4.5x*1.06^{-5} + 5*5x*1.06^{-6} + 6*100*1.06^{-7}$
$= 43.6401x + 399.0343$ [1.5]

Volatility of assets is $-V'[i]_A / V[i]_A$, where $-V'[i]_A$ is
$10*1.06^{-2} + 2*10*1.06^{-3} + 5*20*1.06^{-6} + 6*30*1.06^{-7} + 7*y*1.06^{-8}$
$= 215.8987 + 4.3919y$ [1]

As per condition 2:
$43.6401x + 399.0343 = 215.8987 + 4.3919y$ [0.5]
$x = -4.1965 + 0.100639y$ [0.5]

Using [A],
$0.051305y – 1.2396 = -4.1965 + 0.100639y$
So $y = 59.93$ [1]

Using [A] and value of y from above
$x = 1.84$ [1]

[10]

ii) Convexity of liabilities is $V''_L(i) / V_L(i)$, where $V''_L(i)$ is

$2*1.84*1.06^{-3} + 2*3*2*1.84*1.06^{-4} + 3*4*3.5*1.84*1.06^{-5}$
$+ 4*5*4.5*1.84*1.06^{-6} + 5*6*5*1.84*1.06^{-7} + 6*7*100*1.06^{-8}$ [1]
$= 3013$

Convexity of liabilities = 31.95 [1]

Convexity of assets is $V''_A(i) / V_A(i)$, where $V''_A(i)$ is

$2*10*1.06^{-3} + 2*3*10*1.06^{-4} + 5*6*20*1.06^{-7} + 6*7*30*1.06^{-8} + 7*8*59.94*1.06^{-10}$

$= 3240.6$ [1]
Convexity of assets = 34.37 [1]

Convexity of assets is > convexity of liabilities. Therefore, the portfolio is immunised. [1]

[5]

iii)
1. There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known.
2. By immunising, the possibility of mismatching profits as well as losses is removed apart from a small second-order effect. It also rules out investment in assets with high expected but uncertain returns such as equities and property. (OR immunisation removes the likelihood of making large profits)
3. The theory relies upon small changes in interest rates. The fund may not be protected against large changes.
4. The theory assumes a flat yield curve and requires the same change in interest rates at all terms.
5. In practice the portfolio must be rearranged constantly to maintain the correct balance of:
   (i) equal discounted mean term
   (ii) greater spread of asset proceeds.
   This is because the formulae for duration and convexity both depend upon the times to each payment, which are continuously changing. Note that this is not the case with pure matching, where the asset proceeds emerge as and when required to meet the liability outgo.
6. The theory ignores the dealing costs of a daily (or even monthly) rearrangement of assets.
7. Assets may not exist to provide the necessary overall asset volatility to match the liability volatility.

[4]

[19 Marks]

Solution 2: Option D (Turing test) **[2 Marks]**

Solution 3:

i) Total cost of debt for government would comprise of regular coupon payments i.e. fixed and maturity repayments linked to gold prices prevailing in future i.e. uncertain/unknown. Thus, total cost of debt is also uncertain and unknown to the government.

To estimate the total cost of debt, Government should follow stochastic model

At least one parameter i.e. gold prices on maturity is assigned a probability distribution.

[3]

ii) Steps involved in the modelling process are:
- Develop a well-defined set of objectives that need to be met by the modelling process.
- Plan the modelling process and how the model will be validated.
- Collect and analyse the necessary data for the model.
- Define the parameters for the model and consider appropriate parameter values.
- Define the model initially by capturing the essence of the real world system. Refining the level of detail in the model can come at a later stage.
- Involve experts on the real world system you are trying to imitate so as to get feedback on the validity of the conceptual model.
- Decide on whether a simulation package or a general purpose language is appropriate for the implementation of the model. Choose a statistically reliable random number generator that will perform adequately in the context of the complexity of the model.
- Models based on a deterministic approach would not need this.
- Write the computer program for the model.
- Debug the program to make sure it performs the intended operations in the model definition.
- Test the reasonableness of the output from the model.

- Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.
- Analyse the output from the model.
- Ensure that any relevant professional guidance has been complied with.
- Communicate and document the results and the model.

[5]

iii) The stability of the relationships incorporated in the model may not be realistic in the longer term. For example exponential growth can appear linear if surveyed over a short period of time. If changes can be predicted they can be incorporated in the model, but often it must be accepted that longer-term models are suspect.

Models are by definition, simplified versions of the real world. They may therefore ignore "higher order" relationships which are of little importance in the short term, but which may accumulate in the longer term. [2]

[10 Marks]

Solution 4:

i) Uniform deaths

$${}_{1.75}p_{40.75} = {}_{0.25}p_{40.75} \times p_{41} \times {}_{0.5}p_{42}$$

$$p_{41} = 1 - q_{41} = 1 - 0.001014 = 0.998986 \qquad (0.5)$$

$${}_{0.25}p_{40.75} = 1 - {}_{0.25}q_{40.75} = 1 - \left(\frac{0.25q_{40}}{1-0.25q_{40}}\right) = 1 - \left(\frac{0.25 \times 0.000937}{1 - 0.25 \times .000937}\right)$$

$$= 0.999766 \qquad (0.5)$$

$${}_{0.5}p_{42} = 1 - {}_{0.5}q_{42} = 1 - 0.5q_{42} = 1 - 0.5 \times 0.001104 = 0.999448 \qquad (0.5)$$

Based on the above:

$${}_{1.75}p_{40.75} = 0.999766 \times 0.998986 \times 0.999448 = 0.998201 \qquad (0.5)$$

[2]

ii) Constant force of mortality

$${}_{1.75}p_{40.75} = {}_{0.25}p_{40.75} \times p_{41} \times {}_{0.5}p_{42} \qquad (0.5)$$

$${}_{0.5}p_{42} = (p_{42})^{0.5} = (1 - q_{42})^{0.5} = (1 - 0.001104)^{0.5} = 0.999448 \qquad (0.5)$$

$${}_{0.25}p_{40.75} = \left(p_{40)}\right)^{0.25} = (1 - q_{40})^{0.25} = (1 - 0.000937)^{0.25} = 0.999766 \qquad (0.5)$$

So,

$${}_{1.75}p_{40.75} = 0.999766 \times 0.998986 \times 0.999448 = 0.998200 \qquad (0.5)$$

[2]

[4 Marks]

Solution 5: Option D **[4 Marks]**

Workings only for reference:

The expected present value of maturity benefit is:

$$EPV = 50000 \times \frac{D_{65}}{D_{[45]}} = 50000 \times \frac{689.23}{1677.42} = 20544.35$$

Expected present value of death benefit is:

$$EPV = 1577(IA)^1_{[45]:\overline{20|}} = 1577 \times \left[(IA)_{[45]} - \frac{D_{65}}{D_{[45]}}[(IA)_{65} + 20A_{65}]\right]$$

$$= 1577 \times \left[8.33865 - \frac{689.23}{1677.42}(7.89442 + 20 \times 0.52786)\right]$$

= 1194

So, EPV of benefits = 20,544+1,194 = 21,738 (and nearest value as per options is Option D)

Solution 6: Option A **[4 Marks]**

Workings only for reference:

Let the accumulated value of the investment be X and i_y =investment return for the year y (working in ‘000s)

E(X)=E[4 *(1+i_{2004})*(1+i_{2005})*(1+i_{2006}) +4*(1+i_{2005})*(1+i_{2006})+4*(1+i_{2006})+105+4]

= 4*(1.055*1.06*1.045+1.06*1.045+1.045)+109

=122.29 per 1000

Solution 7: Option B **[2 Marks]**

Workings only for reference:

Return earned = 98/96.5-1 = 1.55%

We note that: (1+4%)^(144/365) = 1.55%. Therefore, Option B.

Q.8) The equation of value for the one-year bond (with one year spot rate y_1) is
97=109/(1+y_1)=> y_1 =12.371%

The equation of value for the two-year bond (with one year spot rate y_2) is

97=6/(1+ y_2) +109/(1+ y_2)^2)

So, $y_{2=}$ 9.143%

[3 Marks]

Solution 9:

i) Loan outstanding on 1 September 2008 = present value of annuity payments

=1000*($v^{10/12}$ +1.05$v^{14/12}$ +1.05^{2} *$v^{18/12}$ +……+1.05^{14} $v^{66/12}$) [1]
=1000*{1-[$v^{4/12}$(1.05)]^{15}}/{1-$v^{4/12}$(1.05)}* $v^{10/12}$ [1]
=17,692 [1]
[3]

ii) Loan outstanding on 1 June 2009 = 17692*$(1.06)^{10/12}$
= 18,572 [0.5]

Interest paid in first instalment = 18,572 – 17,692 = 880 [0.5]

Capital repayment = 1,000 – 880 = 120 [1]

[2]

iii) Capital outstanding after 6 repayments = PV of payment as at 1 March 2011

= 1000 * $(1.05)^6$ *($v^{4/12}$ +1.05*$v^{8/12}$ +….+$(1.05)^8$ *$v^{36/12}$) [1]

= 13,341 [1]

Interest in 7th Payment=13,341 * (($(1.06)^{4/12}$ -1) = 262 [1]
= 262 (0.5)

Capital in 7th payment = 1000 * $(1.05)^6$ – 262 [1]
= 1340 – 262 = 1078 [0.5]

[5]

[10 Marks]

Solution 10:

The present value of the annuity is given by

$$100000\, a^{(12)}_{\overline{60:55}} + 50000\, a^{(12)}_{60:55} + 50000\, a^{(12)}_{60} + 50000\,(v^{10} 10p_{\overline{60:55}} + v^{20} 20p_{\overline{60:55}})$$ [2]

$$a^{(12)}_{60} = \ddot{a}_{60} - \frac{13}{24} = 15.632 - \frac{13}{24} = 15.0903$$ [1]

$$a^{(12)}_{60:55} = \ddot{a}_{60:55} - \frac{13}{24} = 14.756 - \frac{13}{24} = 14.214$$ [1]

$$a^{(12)}_{\overline{60:55}} = \ddot{a}_{60} + \ddot{a}_{55} - \ddot{a}_{60:55} - \frac{13}{24} = 15.632 + 18.210 - 14.214 - \frac{13}{24} = 18.544$$ [1]

$$v^{10} 10p_{\overline{60:55}} = \frac{1 - \left(1 - \frac{l_{70}}{l_{60}}\right)\left(1 - \frac{l_{65}}{l_{55}}\right)}{1.04^{10}}$$ [1]

$$= \frac{1 - \left(1 - \frac{9238.134}{9826.131}\right)\left(1 - \frac{9703.708}{9917.623}\right)}{1.48024} = 0.67469$$ [1]

$$v^{20} 20p_{\overline{60:55}} = \frac{1 - \left(1 - \frac{l_{80}}{l_{60}}\right)\left(1 - \frac{l_{75}}{l_{55}}\right)}{1.04^{20}}$$ [1]

$$= \frac{1 - \left(1 - \frac{6953.536}{9826.131}\right)\left(1 - \frac{8784.955}{9917.623}\right)}{2.19112}$$

$$= 0.44115$$ [1]

So the value is:

$$100000 \times 18.544 + 50000 \times 14.214 + 50000 \times 15.0903 + 50000 \times (0.67469 + 0.44115)$$ [1]

=3375407

[10 Marks]

Solution 11:

The difference between net premium reserve and the gross premium reserve for any conventional without profits contract are:

a) All expenses are ignored; and [0.5]
b) The premium used in the reserve calculation is the net premium, defined below [0.5]

Net premium is calculated:

a) Using prospective calculation approach [0.5]
b) Using the equivalence principle [0.5]
c) Using the same assumptions as the reserve basis [0.5]
d) By not making any allowance for future expenses [0.5]

The notional net premium calculated and valued as the future income element of the reserve is generally considered smaller than the actual premium being paid. [1]
It is considered that the excess of the actual premium over the notional premium will be sufficient to cover the expenses that are not specifically valued [1]

[5 Marks]

Solution 12:

Age of parent = 35 years
Policy term = premium payment term = 13 years
Annual premium = 50,000
Maturity benefit = 900,000

i) To check this, we can calculate the accumulated value of premiums at an interest rate of 4% p.a. and compare with the maturity value [2]

$= 50{,}000 \, * \, \ddot{a}_{\overline{13}} * (1.04)^{13} = 50{,}000 \, * 10.385 * 1.665 = 864{,}596$ [1]

As the accumulated value at 4% is less than maturity value, the return being provided on this policy is more than the policyholders expected return. [1]

Alternatively, the candidate can also work out the IRR on this policy. IRR is 4.55%
As IRR is higher than the expected return, the policy meets the expectations **[4]**

ii) Just after the policyholder attains the age of 43 years, the remaining policy term is 5 years [0.5]

The remaining premium payments are 4 (as these are paid in advance). [0.5]

Reserve = PV maturity benefit + PV expenses + PV commissions – PV premiums [1]

PV maturity benefit = 900,000 * 1.04^{-5} [1.5]
Maturity benefit will be payable whether or not the life insurer survives till the end of the policy term
PV maturity benefit = 739,734 [0.5]

PV expenses = 500 * a_5 [1.5]
The in-force status of policy is independent of the status of the life insured
PV expenses = 500 * 4.45182 = 2,226 [1]

PV premium = 50,000 * $a_{43:4}$ [1]
$a_{43:4} = a_{43} - v^4 * l_{47}/l_{43} * a_{47}$ [0.5]

	= 18.319 – 0.8548* 9771.0789/ 9826.2060 * 17.295 = 3.618102	[0.5]
	PV premium = 180,905	[0.5]
	PV of commissions = 2.5% * PV premiums = 4,523	[1]
	Reserve = 739,734 + 2,226 + 4,523 - 180,905 = 565,578	[1] **[11]**
iii)	The death strain at risk = max(Reserve – death benefit,0) Reserve = 565,578 Death benefit = premiums which are waived on death = 180,905 So death strain at risk = 0	[1] [0.5] [1] [0.5] **[3]**
iv)	If the premium frequency were to increase to monthly, the reserve will reduce.	[1]
	This is because the present value of premiums will increase, as now part of the premium is due in one month rather than after one year	[1] **[2]**
v)	If surrenders are assumed, it is expected that the reserves will reduce.	[1]
	Upon surrender, the benefit payable is maturity value discounted at a rate of 9% p.a.	
	This rate is higher than the reserving interest rate of 4% (or rate that assets are expected to earn on a prudent basis)	
	The money required to fund a surrender is lower than the money required to fund maturity benefit.	[3]
vi)	"Lapse" and "Surrender" are both used to describe events where the policyholder **voluntarily ceases** payment of premium.	[0.5]
	In such an event, when **no cash payment** is made to the policyholder, we refer to it as a "lapse"	[1]
	In the event that policyholder **receives cash value upon termination**, we refer to it as a "surrender"	[0.5] **[2]**
vii)	A multiple decrement model is a **multiple state model**, which has …..one active state; and …... one or more absorbing exit states	[1] [0.5] [0.5] **[2]** **[27 Marks]**

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