Institute of Actuaries of India

Subject CM1-Paper A – Actuarial Mathematics

June 2019 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

In assessing the suitability of a model for a particular exercise, it is important to consider the following:

1. The objectives of the modelling exercise.
2. The validity of the model for the purpose to which it is to be put.
3. The validity of the data to be used.
4. The validity of the assumptions.
5. The possible errors associated with the model or parameters used not being a perfect representation of the real-world situation being modeled.
6. The impact of correlations between the random variables that drive the model.
7. The extent of correlations between the various results produced from the model.
8. The current relevance of models written and used in the past.
9. The credibility of the data input.
10. The credibility of the results output.
11. The dangers of spurious accuracy.
12. The ease with which the model and its results can be communicated.
13. Regulatory requirements.

[Any 6 points – 0.5 for each relevant point]

[3 Marks]

Solution 2:

i) Expectation of an increase in inflation will result in interest rates to increase as lenders will expect interest rates to outstrip inflation. [1]

ii) Increase in tax rates will lead to an increase in interest rates as investors will expect a certain level of return after tax. [1]

iii) Increase in interest rates of a major economy will result in interest rates to increase as investments in the other economy become more attractive reducing demand for local investments. [1]

iv) Sharp increase in number of life insurance companies operating in the market will result in interest rates to fall particularly at the longer terms. This is because liabilities of life insurance companies are longer-dated relative to other industries and thereby increasing demand for longer dated bonds. [2]

[5 Marks]

Solution 3:

i) Duration of an asset: Duration is a measure of interest-sensitivity of an asset. This is the mean term of cash flows, weighted by present value.

Consider a series of cash flows {C_{tk}} for k = 1,2,….,n. Let A be the present value of payments at rate I, so that

$$A = \sum_{k=1}^{n}(C\mathrm{tk} * (1+i)\hat{}(-tk))$$

Then the duration is defined to be =

$$\tau = \sum_{k=1}^{n} (tk * C\text{tk} * (1+i)\hat{}(-tk)) \,/\, A$$

[1]

Effective duration of an asset: This is also a measure of interest-sensitivity of an asset and is defined to be:

$$\nu = \frac{-1}{A}\frac{dA}{di}$$

[1]

Relationship between effective duration and duration:

Effective duration = $\nu = \frac{-1}{A}\frac{dA}{di}$

$= \frac{-1}{A} \frac{d\,(\sum_{k=1}^{n} Ctk*(1+i)\hat{}(-tk))}{di}$

$= \frac{-1}{A} \frac{(\sum_{k=1}^{n} Ctk*(-tk)*(1+i)\hat{}(-tk+1))}{}$

$= \frac{1}{A} \frac{*(1+i)*(\sum_{k=1}^{n} Ctk*tk*(1+i)\hat{}(-tk))}{}$

= **(1+i) * ν**

[2]

[4]

ii) It is given that:

ν = 2.6 and c = 9.7

Given the above, approximation to the change in value of assets for a small change in interest rates (ε) can be determined as

= $-\varepsilon\nu + \varepsilon^2 * ½ * c$

= $-0.5\% * 2.6 + 0.5\%^2 * ½ * 9.7$

= -1.29%

Assets will reduce in value by 1.29%. **[2]**

[6 Marks]

Solution 4:

To find the net premium P:

$P*\ddot{a}_{45:15} = 75{,}000 * A_{45:15} - 25000* D_{60}/D_{45}$ [1]

Hence P = (75000*0.56206 – 25000* 882.85/1677.97)/11.386 [1]

P = INR 2,547.07 [0.5]

To find the retrospective net premium reserve immediately before the sixth premium payment is due:

${}_5V = (P\,\ddot{a}_{45:5} - 75000* A^1_{45:5}) * D_{45}/ D_{50}$ [1.5]

${}_5V = [2547.07 * (\ddot{a}_{45} - D_{50}/ D_{45} * \ddot{a}_{50}) - 75000* (A_{45} - D_{50}/ D_{45} * A_{50})] * D_{45}/ D_{50}$ [1]

$_5V$ = [2547.07 * (18.823 - 1366.61/1677.97 *17.444) – 75000 * (0.27605 - 1366.61/1677.97*0.32907)] * 1677.97 / 1366.61

$_5V$ = (2547.07 * 4.615866 – 75000 * 0.008041) * 1.22783

$_5V$ = INR 13,695.04 [1]

[6 Marks]

Solution 5:

i)

a) $i^{(4)}$ = 6% p.a.
Let i be the effective monthly rate of interest
$(1+i)^{12} = (1+ i^{(4)}/4)^{4}$
i = 0.498%

[1]

b) $i^{(6)}$ = 10% p.a.
Let i be the effective monthly rate of interest
$(1+i)^{12} = (1+ i^{(6)}/6)^{6}$
i = 0.830%

[1]

[2]

ii)

given that

$$\delta(t) = \begin{cases} .05+.005\,t & 0<=t<5 \\ .08t - .01 & 5<=t<10 \\ .10 & 10<=t \end{cases}$$

The present value at time 2 of a payment of 5000 at time 15 years will be

$$PV= 5000*e^{-\int_2^5 (.05+.005t)dt} * e^{-\int_5^{10} (.08t-.01)dt} * e^{-\int_{10}^{15} (.10)dt}$$

[1]

$$= 5000*e^{-[.05t + \frac{.005t^2}{2}]_2^5} * e^{-[\frac{.08t^2}{2} - .01t]_5^{10}} * e^{-[.10t]_{10}^{15}}$$

[1]

$$= 5000*e^{-[.05(5-2) + \frac{.005}{2}(25-4)]} * e^{-[.04(100-25) - .01(10-5)]} * e^{-[.10(15-10)]}$$

[1]

$$= 5000*e^{-[.2025]} * e^{-[2.95]} * e^{-[.5]}$$

= 129.63

[1]

Hence the present value is Rs. 129.63

[4]

[6 Marks]

Solution 6:

(i)

- The expression suggests that this is an endowment assurance contract, with benefit on death and maturity being equal to S and term of contract being n years. [1]
- Death benefits are assumed to be payable immediately on death
- Maturity benefit is paid at the end of the contract term
- Premiums are level and are payable continuously
- Initial expenses are incurred on the day the contract is sold and are assumed to be a one-time cost
- Renewal expenses are also assumed to be payable continuously
- Claim related expenses are assumed to be payable immediately on death and on maturity.

[0.5 mark for each valid point]

[4]

(ii)

The reserve required to be held at the end of time 2 =

$110{,}000 * (v * q_{49} + v^2 * p_{49} * q_{50} + v^3 * {}_2p_{49} * q_{51}) – 2000 * (1 + v* p_{49} + v^2\ {}_2p_{49})$

[1]

We now need to calculate each of the above probabilities.

$p_{49} = e^{(-0.025)} = 0.97531$

$p_{50} = e^{(-0.03)} = 0.970446$

$p_{51} = e^{(-0.03)} = 0.970446$

${}_2p_{49} = p_{49} * p_{50} = 0.9465$

$v = 0.94340, v^2 = 0.89, v^3 = 0.83962$ [2]

Reserve =

110000 * (0.94340 * (1 - 0.97531) + 0.89 * 0.97531 * (1 - 0.970446) + 0.83962 * 0.9465 * (1 - 0.970446) – 2000 * (1 + 0.94340 * 0.97531 + 0.89 * 0.9465)

= 110000 * 0.07243 – 2000 * 2.762

=7967.3 – 5525

=2442 [1]

[4]

[8 Marks]

Solution 7:

(i) (a) Endowment Assurance

The endowment assurance, in return for a series of regular or single premium, provides following benefits:

- a survival benefit at the end of the term
- a lump sum benefit on death if it occurs before the end of the term.

Cash flows from policyholder's perspective, it will be a series of negative cash flows throughout the specified term or until death if earlier followed by a large positive cash flow at the end of the term (or death if earlier)

Cash flows from insurer's perspective, there is a stream of regular positive cash flows which cease at a specified point (or earlier if the policyholder dies), followed by a large negative cash flow. The negative cash flow is certain to be paid but the timing of that payment depends on whether / when the policyholder dies [2]

(b) Term assurance

A term assurance is an insurance policy which provides a lump sum benefit on death before the end of the specified term usually in return for a series of regular premiums.

Cash flows from policyholder's perspective, it will be a series of negative cash flows throughout the specified term or until death if earlier followed by a large positive cash flow payable on death, if death occurs before the end of the term. If the policyholder survives to the end of the term there is no positive cash flow.

Cash flows from insurer's perspective, there is a stream of regular positive cash flows which cease at a specified point (or earlier, if the policyholder dies), followed by a large negative cash flow, contingent on policyholder death during the term.

[2]

[4]

Note: The solution is exhaustive and is for reference purpose. If a candidate has mentioned about the cash flows correctly, not necessarily from both the insurer's and policyholder's perspective, full marks should be awarded.

(ii) For a fixed purchase amount, the annuity amount is inversely proportional to the expected number of annuity instalments, i.e. if the number of instalments is expected to be more, it reduces the annuity amount.

The option (c) is expected to give the lowest number of annuity instalments as it will commence on the death of the first life and will be payable till the survival of the second life. Further, in other options the number of annuity instalments payable is expected to be more than under option (c). The lowest annuity amount will be for option (b) as the payments are expected to be for longer period. Hence option (c) will give highest annuity amount. **[2]**

(iii) a < c < e < d < b

[2]

[8 Marks]

<u>Solution 8:</u>

Let P be the level annual premium. The expected PV of the premium is:

$P^* \ddot{a}_{60:56:5} = P * (\ddot{a}_{60:56} - v^5 *{}_5p_{56} *{}_5p_{60} * \ddot{a}_{65:61})$ [0.5]

Where the 60 year old is subject to male mortality rates and the 56 year old is subject to female mortality rates.

Using the tables:

$P* \ddot{a}_{60:56:5} = P * (14.646 – 1.04^{-5} * 9828.163/ 9907.249 * 9647.797/9826.131 * 12.560)$

$= 4.59087 P$ [1]

The expected present value of the expenses is:

$500 + 0.02 * P * (\ddot{a}_{60:56:5} -1) = 500 + 0.02*P*3.59087 = 500 + 0.07182*P$ [0.5]

The expected present value of the benefit payable on death is 100,000 for the first 5 years and 200,000 between 6 to 10 years.

$100,000 * A^1_{60:56:10} + 100,000 A^1_{65:61:5} * v^5 * {}_5p_{56} * {}_5p_{60}$ [1]

where

$A^1_{60:56:10} = (A_{60:56} – v^{10} * {}_{10}p_{56} * {}_{10}p_{60} A_{70:66})$ [0.5]

$= (1-d* \ddot{a}_{60:56} – v^{10} * {}_{10}p_{56} * {}_{10}p_{60} * (1-d* \ddot{a}_{70:66}))$

$=(1- 0.04/1.04 * 14.646 – 1.04^{-10} * 9658.285/9907.249 * 9238.134/9826.131*(1-0.04/1.04 * 10.368))$

$=0.43669 – 0.61918 * 0.60123$

$=0.06442$ [0.5]

Similarly,

$A^1_{65:61:5} = = (A_{65:61} – v^5 * {}_5p_{61} * {}_5p_{65} A_{70:66})$ [0.5]

$= (1-d* \ddot{a}_{65:61} – v^5 * {}_5p_{61} * {}_5p_{65} * (1-d* \ddot{a}_{70:66}))$

$=(1- 0.04/1.04 *12.560 – 1.04^{-5} * 9658.285/9828.163 * 9238.134/9647.797*(1-0.04/1.04 * 10.368))$

$= 0.51692 – 0.77342 *0.60123$ [0.5]

$= 0.05192$

Total EPV of lumpsum benefit:

$100,000 * A^1_{60:56:10} + 100,000 A^1_{65:61:5} * v^5 * {}_5p_{56} * {}_5p_{60}$ [0.5]

$= 100,000 * 0.06442 + 100,000 * 0.05192* 1.04^{-5} * 9828.163/ 9907.249 * 9647.797/9826.131$

=INR 10,598.55

The deferred annuity of INR 20,000 p.a. while both lives are alive or INR 10,000 if only one of them is alive. This is equivalent to each life receiving a single life annuity of INR 10,000 (since 20,000 will be paid in total if both are alive) [0.5]

$10,000 * (v^{10} * {}_{10}p_{56} * \ddot{a}_{66} + v^{10} * {}_{10}p_{60} * \ddot{a}_{70})$

$= 10,000 * 1.04^{-10} * (9658.285/9907.249*14.494 + 9238.134/9826.131*11.562)$

= INR 168,890.38 [1]

Setting the EPV of premiums equal to the EPV of benefits and expenses gives:

4.59087 P = 168,890.38 + 10,598.55 + 500 + 0.07182*P [0.5]

4.51905P = 179,988.93

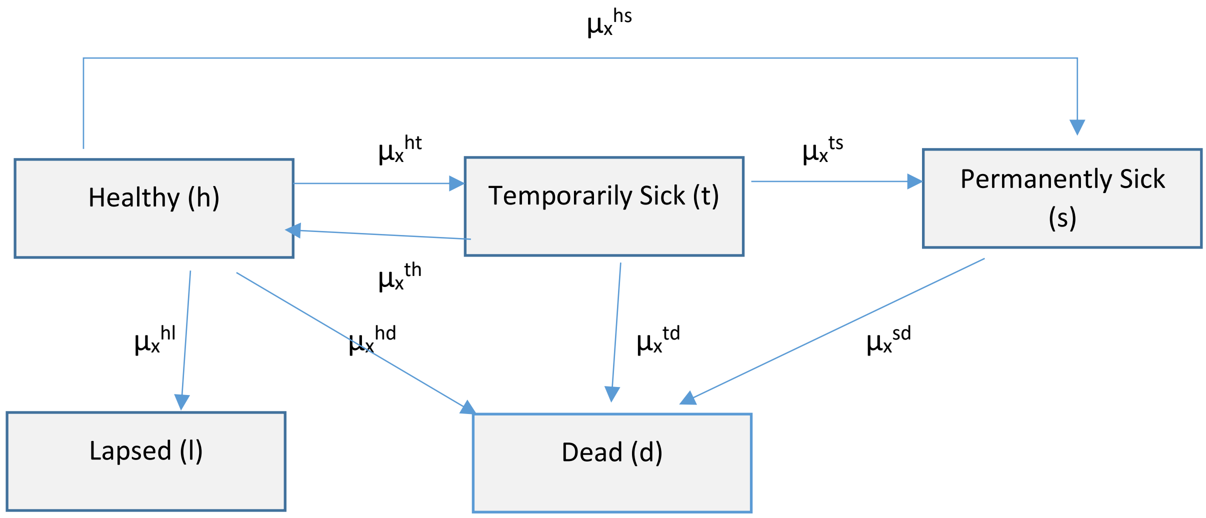
P = INR 39,828.93 [0.5]

[8 Marks]

Solution 9:

i) *Three key states to include will be healthy, sick and dead. However, the sick states would be either of the two: a temporarily (recoverable) sick state and a permanent (non-recoverable) sick state as different benefit levels are paid in each. It would also be sensible to include lapse, as these policies are paid for by annual premiums so they can be lapsed if policyholders stop paying their premiums early.*

We also need to include the transitions where there are changes in payments.



[5]

ii) EPV(premiums) = EPV(benefits) [1]

EPV of temporary sickness benefit:

$10{,}000\int_{r=0}^{15} v^r \, {}_rp_{50}^{ht} \, dr$ [1]

EPV of permanent sickness benefit:

$20{,}000\int_{r=0}^{15} v^r \, {}_rp_{50}^{hs} \, dr$ [0.5]

EPV of death benefit: On death at time r, a benefit of INR 1,00,000 would be payable, regardless of which state is then occupied

$100{,}000\int_{r=0}^{15} v^r ({}_rp_{50}^{hh}\, \mu_{50+r}^{hd} + {}_rp_{50}^{ht}\, \mu_{50+r}^{td} + {}_rp_{50}^{hs}\, \mu_{50+r}^{sd})dr$ [1.5]

EPV of premium:

Premiums are paid by healthy lives. So the expected present value is:

$P\int_{r=0}^{15} v^r \, {}_rp_{50}^{hh} \, dr$ [1]

So by setting EPV of benefits = EPV of premium, we obtain:

P = {10,000$\int_{r=0}^{15} v^r {}_rp_{50}^{ht}$ dr + 20,000$\int_{r=0}^{15} v^r {}_rp_{50}^{hs}$ dr + 100,000$\int_{r=0}^{15} v^r ({}_rp_{50}^{hh}\, \mu_{50+r}^{hd} + {}_rp_{50}^{ht}$ $\mu_{50+r}^{td} + {}_rp_{50}^{hs}\, \mu_{50+r}^{sd}$)dr} / {$\int_{r=0}^{15} v^r {}_rp_{50}^{hh}$ dr}

[2]

[7]

[12 Marks]

Solution 10:

i) $i^{(2)}$ = 8% p.a.

Let i be the effective annual rate of interest

(1+i) = (1+ $i^{(2)}$/2)^2

i= 8.16% p.a.

Let i_{12} and i_4 be the effective monthly and quarterly rate of interest

i_{12} = (1+i)^(1/12)-1 = 0.66%

i_4 = (1+i)^(1/4)-1 = 1.98%

[0.5]

Outgo:

- At outset = Rs 15,00,000
- Maintenance cost:
  Payable for 9 years
  Maintenance cost per quarter = 5%* 15,00,000/4 = 18,750

[0.5]

PV of maintenance cost = $v^{@i}$ * 18750* $\ddot{a}_{\overline{36}|}$@ i_4

[1]

Total outgo =
15,00,000 + $v^{@i}$ * 18750* $\ddot{a}_{\overline{36}|}$@ i_4
=15,00,000 + 4,51,956
=19,51,956

[1]

Income:

- Buyback price:

  Depreciation (D) = 15,00,000*2%*10 = 3,00,000
  Buyback price at the end of 10th year = 15,00,000-3,00,000 = 12,00,000

[0.5]

PV of buyback price at outset = 12,00,000 * v^{10} @i
= 5,47,664

[1]

- Rental income:

Monthly rent = 2,40,000/12 = 20,000

PV of rental income = 20,000 * $a_{120]}$@ i_{12}= 16,57,813

[1]

Total income = 5,47,664 + 16,57,813 = 22,05,477

NPV = Income - Outgo

= 2,53,521 [0.5]

[6]

ii) NPV with Second option

Outgo:

PV of lease rent = (180000 / 12) * $a_{120]}$@ i_{12}

= 12,43,359

[1]

Investment amount = 15,00,000

Income:

PV of income = 15,00,000*(1.10)^10 / (1.0816)^10

= 17,75,625

[1]

NPV = Income – Outgo – Investment Income

= 17,75,625 - 12,43,359 – 15,00,000 = -9,67,734

[0.5]

The first option is more profitable as its NPV is more as compared to first option.

[0.5]

[3]

[9 Marks]

Solution 11:

i) 4 areas that can be expected to be part of the bonus philosophy of a company are:

1. What form bonuses take (i.e. simple reversionary, compound reversionary, terminal etc.)
2. What proportion of surplus the company distributes to policyholders.
3. What degree of smoothing the company operates in respect of passing investment profits to policyholders.
4. Broad investment strategy of the company.

[1 mark for each point above (or any other suitable example not listed above)]

[4]

ii) Pricing basis: The set of assumption used by an insurance company to calculate a premium is called the pricing basis. **[1]**

iii) Let Single premium = P

Monthly Annuity amount (S) = 10,000

Initial expense (f) = 2% of Single Premium

Regular expense (e)= 0.5% * 12 * S

Claim expenses = 0.5% * P at the time of death

Age of policyholder (x) = 65

Annuity payments and regular expenses are assumed to be made in arrears. Claim expenses are paid immediately upon death.

By principle of equivalence:

$P = 12 * S * a_{65}^{(12)} + 2\% * P + 0.5\% * 12 * S * a_{65} + P\ \bar{A}^1_{65:10} + 0.5\% * P * \bar{A}^1_{65:10}$

[1.5]

(Full marks can be awarded to the candidate if regular expenses are considered as monthly in the above formula)

In the equation above:

$a_{65}^{(12)} = \ddot{a}_{65}^{(12)} - 1/12 = \ddot{a}_{65} - (12-1)/(2*12) - 1/12 = 13.666 - 11/24 - 1/12 = 13.124$

$a_{65} = \ddot{a}_{65} - 1 = 13.666 - 1 = 12.666$

[1]

$\bar{A}^1_{65:10} = 1.04^{(1/2)} * A^1_{65:10}$

$A^1_{65:10} = A_{65} - 1.04^{(-10)} * {}_{10}p_{65} A_{75}$

$A_{65} = 1 - 0.04/1.04 * \ddot{a}_{65} = 1 - 0.04/1.04 * 13.666 = 0.474385$

$A_{75} = 1 - 0.04/1.04 * \ddot{a}_{75} = 1 - 0.04/1.04 * 9.456 = 0.636308$

${}_{10}p_{65} = l_{75} / l_{65} = 8405.160 / 9647.797 = 0.8712$

So $\bar{A}^1_{65:10} = 1.04^{(1/2)} * (0.474385 - 1.04^{(-10)} * 0.8712 * 0.636308) = 0.101863$ [2.5]

Therefore,

P = 120000 * 13.124 + 0.02 * P + 600 * 12.666 + P * (0.101863 + 0.005 * 0.101863)

P (1 – 0.02 – 0.102372) = 1,582,480

P = 1,803,133

[1]

[6]

iv)

a. If death benefit payable at end of year of death, this will reduce the present value of death benefit to be paid out. This will also reduce the present value of claim related expenses which are assumed to be incurred at the time of death payment. Therefore, the single premium will reduce. [1]

b. If annuity is payable annually in arrears then PV of annuity outgo will reduce, as roughly $a_{65}^{(12)} = a_{65} + 11/24$, which means that $a_{65}^{(12)}$ is higher than a_{65}. Therefore, single premium will reduce. [1]

[13 Marks]

Solution 12:

(i) $i^{(12)} = 10\%$ p.a.

Let i be the effective monthly rate of interest

i= 10% / 12 = 0.83%

Let monthly instalment be X

35,00,000 = X $a_{240\rceil}^{@\ i}$

X = 33, 683 per month [2]

(ii) Outstanding loan at the end of 5 years

= 33683 * $a_{180\rceil}^{@\ i}$

[1]

= 33683 *93.26972

=31,41,604 [0.5]

Repayment = 5,00,000

Loan after repayment = 31,41,604 – 5,00,000 = 26,41,604 [0.5]

Let revised monthly instalment be X'

X' $a_{180\rceil}^{@\ i}$= 26,41,604

X' = 28,322 [1]

[3]

(iii) Let t be the number of months after 5 years when the outstanding loan after partial repayment will be paid provided the monthly instalment (X) is maintained as calculated in (i)

X $a_{t\rceil}^{@\ i}$ >= 26,41,604

[1]

$a_{t\rceil}^{@\ i}$ = 26,41604 / 33683 = 78.42544

or v^t = 0.349069

[1]

which gives t = 127.33 months i.e. another 128 months will be required to repay the loan, i.e. 128 months is the outstanding duration [1]

[3]

(iv) Employee chooses to go with option (ii) i.e. monthly instalment same as calculated in (ii)

At the end of 5 years and post partial repayment, the monthly instalment is 28,322 for remaining 15 years of the loan

On 1st Jan 2019, the outstanding loan amount is

= 28,322 * $a_{132\rceil}^{@\ i}$

= 22,66,258

[1]

The personal loan amount is thus Rs 22,66,258 @ 7 % p.a. convertible monthly.

The monthly effective rate of interest is thus

i' = .07/12 = 0.58%

Let t’ be the remaining months after 1st Jan 2019 when the loan will be repaid assuming instalment of X’ (i.e. 28,322)

28,322 * $a_{t'⌉}$@ I’ >= 22,66,258

[1]

$a_{t'⌉}$@ I’ = 22,66,258 / 28,322 = 80.01758

or $v^{t'}$ = 0.535898

[1]

which gives t’ = 107.86 months, i.e. another 108 months will be required to repay the loan.

[1]

[4]

(v) Total interest paid over the term of the loan:

Interest payment in first 5 years:

= total monthly instalment amount in 5 years less loan repaid in first 5 years

= 33,683*60 – [35,00,000 – 31,41,604]

= 16,62,584 --------------------------- (A)

[1.5]

Interest payment in next 4 years:

=28,322*48 – [26,41,604 – 22,66,258]

= 9,84,410 ----------------------------(B)

[1]

Interest payment from end of 9 years till repayment using personal loan:

Balance number of months = 108

Interest repayment = 28,322*108 – 22,66,258

= 7,92,518 -----------------(C)

[1]

Total interest repayment:

= A + B + C

=34,39,212 [0.5]

[4]

Note: - above calculations reflect only unrounded numbers. Candidate’s response in either rounded or unrounded basis should be awarded fully.

Alternatively, if the student does these calculations considering 0.8333% as effective monthly interest rates then the solution for different parts would be:

i) EMI amount = 33,775
ii) Revised EMI Amount = 28,402
iii) Revised tenure = 127.23 or 128 months
iv) Revised tenure = 107.84 or 108 months, using 0.5833% as new effective rate.
v) Total Interest = 34,57,212

[16 Marks]

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