Institute of Actuaries of India

Subject CM2A – Financial Engineering and Loss Reserving (Paper A)

November 2019 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) The market portfolio is (2/7, 3/7, 2/7), so
RM = (2*RA* + 3*RB* + 2*RC*) / 7. [1]

Thus
Cov(*Ri*, *RM*) = [2 Cov(*Ri*, *RA*) + 3Cov(*Ri*, *RB*) + 2 Cov(*Ri*, *RC*)] / 7 [1]

So,
Cov(*RA*, *RM*) = [2 * 0.3^2+ 3*0.5*0.3*0.2+ 2*0.5*0.3*0.1] / 7 = 0.042857143 [1]

Similarly,
Cov(*RB*, *RM*) = 0.028571 , [1]
Cov(*RC*, *RM*) = 0.011429 [1]
and
Var(M) = [2 Cov(*RM*, *RA*) + 3 Cov(*RM*, *RB*) + 2 Cov(*RM*, *RC*)] / 7 = 0.027755 [1]

We conclude that β*A* = 1.5441, β*B* = 1.0294 and β*C* = 0.4118. [1.5]

EM = 12%, Ro= 7%
Finally, solving
Ri - R0 = β*i** (*EM* - *R*0), we get *RA* = 14.7206% , *RB* = 12.1471% and *RC* = 9.0588% [1.5]
[9]

ii) Using $V_i = \beta^2_i \, V_M + V_{\varepsilon i}$ in which the first term of RHS is Systematic risk & the second Specific risk [1]

Company	Specific	Systematic	Vi
A	0.02382	0.06618	0.09
B	0.01059	0.02941	0.04
C	0.00529	0.00471	0.01

[3]
[4]
[13 Marks]

Solution 2:

i) Lundberg's inequality states that

$$\psi(U) \le \exp\{-RU\}$$

where *U* is the insurer's initial surplus and ψ(*U*) is the probability of ultimate ruin. *R* is a parameter associated with a surplus process known as the adjustment coefficient. Its value depends upon the distribution of aggregate claims and on the rate of premium income. [2]

ii) **a)** $f(x) = 2e^{-2x}$
The Moment Generating Function (MGF) is given by

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$ [1]

$$= \int_0^\infty e^{tx} * 2e^{-2x} dx$$

$$= 2\int_0^\infty e^{(t-2)x} dx$$

$$= \frac{2}{(t-2)}[e^{(t-2)\alpha} - e^{(t-2)0}]$$

[1]

$$= \frac{2}{(2-t)}$$

The integral converges for t <2, and hence MGF is valid for t <2. [1]
[3]

b) For the Adjustment co-efficient, we require
$1+(1+\theta)E(x)r = M_x(r)$ [1]

From the solution in Part (B), we have

$$M_x(t) = \frac{2}{(2-t)}$$

$$M_x(r) = \frac{2}{(2-r)}$$

Given that $\Theta = 1/3$

$$M'_x(t) = \frac{2}{(2-t)^\wedge 2}$$

$$M'_x(0)=1/2 = E(x)$$

[1]

Substituting all the values in the equation of Adjustment co-efficient,

$$1+(1+1/3)*1/2*r = \frac{2}{(2-r)}$$

Or, $1+2/3*r = \frac{2}{(2-r)}$

Or, $(3+2r)/3 = \frac{2}{(2-r)}$

Or, $6-3r+4r-2r^\wedge2 = 6$

Or, $r - 2r^\wedge 2 = 0$

Or, $r=0,\ 1/2$

The only positive solution of the adjustment co-efficient is r = ½ [2]

[4]

[9 Marks]

Solution 3: Let $dXt = At\ dt + Bt\ dZt$,

Where, $At = \alpha\ \mu\ (T\text{-}t)$, $Bt = \sigma\sqrt{(T-t)}$ Eq 1 [1]

$dF = \frac{\partial f}{\partial x} Bt\ dZt\ + \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} At + \frac{1}{2}\frac{\partial 2f}{\partial x2} Bt\right) dt$ *(Ito's lemma)* [1]

$= -f\,Bt\ dZt + f\frac{\partial m}{\partial t}(T-t)dt\ \ - f\ At\ dt + \frac{1}{2}\ f\ Bt^2\ dt$

[Since $\frac{\partial f}{\partial x} = -e^{(m(T-t)-x)}$ and $\frac{\partial f}{\partial t} = \frac{\partial m}{\partial t}(T-t) * e^{(m(T-t)-x)}$
(using chain rule) and $\frac{\partial 2f}{\partial x2} = e^{(m(T-t)-x)}$] [2]

$dF = f\left(\frac{\partial m}{\partial t}(T-t) - At + \frac{1}{2}Bt^2\right) dt - f\,Bt\ dZt$ [0.5]

For f to be a martingale, $\frac{\partial m}{\partial t}(T-t) - At + \frac{1}{2}Bt^2 = 0$ [1]

Thus, $\frac{\partial m}{\partial t}(T-t) = At - \frac{1}{2}Bt^2$
Substituting Eq 1 above gives $\frac{\partial m}{\partial t} = \alpha\ \mu - \frac{1}{2}\sigma^2$ [0.5]

[6 Marks]

Solution 4:
i) Returns and solving for L and D:

For Luckworth's offer, returns are:
- \$(100-L) if coin toss outcome predicted correctly (0.5 probability)
- -\$L (i.e. loss of \$L) if coin toss outcome predicted incorrectly (0.5 probability) [0.5]

This being a fair gamble, the expected return should be zero. Therefore, 0.5 * (100-L) + 0.5 * (-L) = 0.

Solving for L yields L = \$50 [0.5]
For Dewis's offer, returns are: \$(100s-D) where s is a random number generated uniformly from the interval [0,1]. [0.5]

This being a fair gamble, the expected return should be zero. E(100s-D) = 0 => 100*E(s) – D = 0.
E(s) = 0.5 as s is generated uniformly from [0,1]. Solving for D yields D = \$50. [0.5]
[2]

ii) <u>Risk measures:</u>

a. <u>Variance</u>

For both L and D, Variance = $E[(X-\mu)^2] = E(X^2)$ since the mean return is zero (as both gambles are fair) [1]
For L, variance = $0.5*50^2 + 0.5*(-50)^2 = 2500$ [0.5]
For D, variance = $\int_0^1 (100s-50)^2 ds$ = 2500/3 = 833.33 (approx) [1]

b. <u>Semi-variance</u>

Downside semi-variance for a random variable X with mean μ is defined as:

$$\int_{-\infty}^{\mu} (\mu - x)^2 f(x) dx$$ for a continuous random variable (such as return for D)

$$\sum_{x<\mu} (\mu - x)^2 P(X = x)$$ for a discrete random variable (such as return for L) [1]

For L, downside semi-variance = $0.5*(-50)^2 = 1250$ [0.5]

For D, variance = $\int_0^{0.5} (100s-50)^2 ds$ = 1250/3 = 416.67 (approx) [1]

c. <u>90% Value-at-Risk (VaR):</u>

90% VaR indicates the maximum level of loss with a 90% confidence, i.e. there is a 10% probability of a greater loss. [0.5]

For L, since there is a 50% probability each of a \$50 loss and a \$50 profit, the 90% VaR will be \$50. [1]

For D, VaR = -t such that P(100s-50<t) = 0.1 => P(s<(t+50)/100) = 0.1 => (t+50)/100 = 0.1 => t = -40. Therefore, 90% VaR is \$40. [1]

d. <u>Expected Shortfall (ES) at -\$10 threshold:</u>

For a random variable X and a threshold level K, Expected Shortfall = E[max(K-X,0)] [0.5]

For L, since there is a 50% probability each of a \$50 loss and a \$50 profit, the ES will be 0.5*(-10-(-50)) = \$20 [1]

For D, loss threshold K = -\$10 corresponds to s = 0.4 (obtained by solving 100s – 50 = -10). Therefore, ES will be $\int_0^{0.4} (-10-(100s-50)) ds = \$8$. [1]
[10]

iii) Comments:

- Using all risk metrics, Luckworth's gamble is consistently riskier than Dewis's from the gambler's perspective even though both have the same expected return (i.e. \$0) and range of returns (i.e. -\$50 to \$50).
- This is because Luckworth's gamble only allows for two extreme outcomes (\$50 profit and loss with equal probabilities), while Dewis's gamble allows for a spectrum of intermediate outcomes
- Since both the gambles have symmetric payoffs around zero, the downside semi-variance is exactly half of the total variance in both cases

[1 for each point, due credit for other comments] [Max 2]

iv) Risk preferences:

- Both gambles are fair but risky. Since Steven doesn't choose either gamble, he should be risk-averse.
- Frank and Tony have opted for one of these gambles, so both of them should be risk-seeking.
- Since Frank has gone for the more risky option (Luckworth's gamble) of the two, he should be more risk-seeking than Tony.

[1 for each point, due credit for other comments] [Max 3]

[18 Marks]

Solution 5:

i) Meaning of equation:

The given equation provides:

- a general formula for find the fair price V_t at any time t of a derivative contract
- having only a terminal payoff X_T at maturity time T

The fair price is arrived at by:

- computing the expectation under the risk-neutral measure Q of the terminal payoff…
- … conditional on the filtration F_t (available information at time t) and then…
- … discounting it using the continuously compounding risk-free rate r for rest of the term (T-t)

[0.5 for each point] [Max 2]

ii) Derivation of digital call option price:

The payoff function for the digital call option is:

$$X_{dc} = \begin{cases} 1 & if\ S_T \geq K \\ 0 & if\ S_T < K \end{cases}$$ [0.5]

To derive the pricing formula, we substitute the payoff in the expression in previous part and assume current time t = 0.

$$V_{dc} = e^{-rT} E_Q[X_{dc}|F_0] = e^{-rT} \int_K^{\infty} 1 \cdot f(S_T|S_0) dS_T$$

where $f(S_T|S_0)$ is the probability density function of S_T (share price at expiry), given S_0 (current share price) under the risk neutral measure. [1]

The distribution of $S_T|S_0$ is lognormal. Also, under the risk neutral measure, the price for a non dividend paying stock should grow at the risk free rate r. Thus,

$$S_T|S_0 \sim \log N[\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T]$$ [1]

We can compute V_{dc} by evaluating the integral using the formulae from the Tables by substituting / replacing L, U, μ, σ^2, k with K, ∞, $\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T$, $\sigma^2 T$, 0 respectively.

$$V_{dc} = e^{-rT}[e^0][\Phi(U_0) - \Phi(L_0)]$$

[1]

where

$$U_0 = \frac{\ln(\infty) - \cdots}{\cdots} - 0 = \infty$$ [0.5]

and

$$L_0 = \frac{\ln(\mathrm{K}) - \{\ln(\mathrm{S}_0 + \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)\mathrm{T}\}}{\sigma\sqrt{\mathrm{T}}} - 0 = -d_2$$ [1]

Therefore,

$$V_{dc} = e^{-rT}[1][\Phi(\infty) - \Phi(-d_2)] = e^{-rT}[1 - \Phi(-d_2)] = e^{-rT}\Phi(d_2)$$ [1]

[6]

iii) Derivation of digital range option price:

The payoff function for the digital range option is:

$$X_{dr} = \begin{cases} 1 \; if \; K_U > S_T \geq K_L \\ 0 \quad\quad otherwise \end{cases}$$ [0.5]

We observe that this payoff can be expressed in terms of payoffs for digital call options as:

$$X_{dr} = X_{dc}(with\ strike\ K_L) - X_{dc}(with\ strike\ K_U)$$ [1]

This can be verified as follows:

S_T	*Digital range*	*Digital call (with strike K_L): M*	*Digital call (with strike K_U): N*	*M-N*
$S_T \geq K_U$	*0*	*1*	*1*	*0*
$K_U > S_T \geq K_L$	*1*	*1*	*0*	*1*
$S_T < K_U$	*0*	*0*	*0*	*0*

Therefore, the digital range option price can be expressed as:

$$V_{dr} = V_{dc}(with\ strike\ K_L) - V_{dc}(with\ strike\ K_U)$$

$$V_{dr} = e^{-rT}\Phi(d_{2L}) - e^{-rT}\Phi(d_{2U})$$

where d_{2L} and d_{2U} are the usual expressions for d_2 with K replaced with K_L and K_U respectively [0.5]

[2]

iv) Relationship between digital call and put option prices:

The payoff function for the digital put option is:

$$X_{dp} = \begin{cases} 1 \; if \; K > S_T \\ 0 \; otherwise \end{cases}$$ [0.5]

We observe that the sum of the payoffs of digital call and digital put options always adds up to 1:

$$X_{dc} + X_{dp} = 1$$ [1]

This can be verified as follows:

S_T	Digital call (with strike K): M	Digital put (with strike K): N	M+N
$S_T \geq K$	1	0	1
$S_T < K$	0	1	1

Given that all payoffs are at time T, the relationship in their current price can be expressed as:

$$V_{dc} + V_{dp} = Present\ value\ of\ 1\ paid\ at\ time\ T$$
$$V_{dc} + V_{dp} = e^{-rT}$$ [0.5]

[2]

v) Advantages of martingale approach compared to PDE approach:

- The main advantage of the martingale approach is that it gives us much more clarity in the process of pricing derivatives
  - Under the PDE approach, one has to 'guess' the solution for a given set of boundary conditions
  - Under the martingale approach, there is an expectation which can be evaluated either explicitly in some cases or numerically in other cases

[1 for each sub-point]

- The martingale approach also gives us the replicating strategy for the derivative
- The martingale approach can be applied to any FT-measurable derivative payment, including path dependent options (such as Asian options), whereas the PDE approach cannot always be

[0.5 for each] **[3]**

vi) Why is PDE approach used sometimes?

PDE approach is sometimes used as it is:
- Much quicker and easier to construct
- More easily understood

[2]

[17 Marks]

Solution 6: Using black scholes to calculate the value of a call option based on the value of the

assets exceeding a strike of L = 21.28601 (15 *$e^{5*.07}$) crores after 5 years.

So = 20 crores, r = .07 , σ=.4 , (T-t) = 5 . [2]

d 1 = 0.768852, Φ (d 1) = 0.779009 [1]

d 2 = -0.12558 , Φ (d 2) = 0.450034 [1]

Value of equity of company B = E = 20 * Φ (d 1) - 15* Φ (d 2) = 8.829677 crores

Value of bond is B = 20 – E = 11.17032 crores [1]

Solving e^{-5*x}= 11.17032 / 21.28601 [1]

$\Rightarrow x = 12.8958\%$

Now, credit spread of Company 'A' = 13.8958% , term = 7 years

Value of 100 nominal is $e^{-7*.138958}*100$ = Rs 37.806 [2]

[8 Marks]

Solution 7:

i) Utility functions having property of constant relative risk aversion are said to be iso-elastic functions. [1]

ii)

$$U(x) = (x^{5\alpha} - 1)/10\alpha$$
$$U'(x) = 5\alpha * x^{5\alpha - 1} / 10\alpha$$
$$= 0.5 * x^{5\alpha - 1}$$

$$U''(x) = 0.5*(5\alpha - 1)* x^{5\alpha - 2}$$ [1]

Thus for the function to satisfy the principle of non-satiation and diminishing marginal utility of wealth, we require $\alpha < 1/5$ [0.5]

The absolute risk aversion is given by

$$A(x) = -U''(x) / U'(x) = -(5\alpha - 1) / x$$

$$A'(x) = (5\alpha - 1) / x^2 < 0$$ [1]

The relative risk aversion is given by

$$R(x) = x*-U''(x) / U'(x)$$
$$= -(5\alpha - 1)$$

$$R'(x) = 0$$ [1]

Thus the function exhibits the property of declining absolute risk aversion and constant relative risk aversion. [0.5]
Hence, the function is iso-elastic.

[4]

iii) The utility function of the individual is given by

$$U(x) = Log(x)$$

Let p be the insurance premium that the individual is willing to pay to protect against the Random Loss of x.
Therefore, we can write
E[U(a – x)] = U(a – p), where a is the initial wealth of the individual. [1]

We have,
X = 500, a = 1000
E[U(a – x)]= 0.5*Log(1000-500) + 0.5*Log1000
=0.5*6.21 + 0.5 *6.91
=6.56 [1]

Using the above equation, we have
Log(1000 – p) = 6.56

Or, (1000 – p) = 707.11

Or, p = 292.89 [2]

Insurance premium for the individual is 292.89.

[4]

[9 Marks]

Solution 8:

i) Premium charged for the policy for 12 months = 15,000

Earned premium = (12-4)/12 *15,000 = 10,000 [1]

Incurred claims = 3,000

Loss ratio = (Incurred claims/Earned premium)
= 3,000 / 10,000
= 30% [1]

[2]

ii) The cumulative claims data for each year is given by:

		Development Year				
		1	2	3	4	5
Accident Year	1	25,000	40,000	50,000	58,000	65,800
	2	27,000	50,000	70,000	89,000	-
	3	29,000	55,000	78,400	-	-
	4	31,500	60,400	-	-	-
	5	33,000	-	-	-	-

[1]

The ultimate loss ratio is given by 65,800 / 70,000 = 0.940
The initial ultimate liability at the end of each year is given by:
2: 0.940*83,000 = 78, 020
3: 0.940 * 91,000 = 85, 540
4: 0.940 * 102,000 = 95, 880
5: 0.940* 110,000= 103, 400 [1]

The development factors are:
Year 4 to 5: (65, 800 / 58, 000) = 1.134
Year 3 to 4: (58,000+89,000)/(50,000+70,000) = 1.225
Year 2 to 3: 1.368
Year 1 to 2: 1.826 [2]

The emerging liabilities for each year are:
2: 78, 020* (1 – 1/1.134) = 9,249
3: 85, 540* (1 – 1/(1.134*1.225)) = 23,989
4: 95, 880* (1 – 1/(1.134*1.225*1.368)) = 45, 458
5: 103, 400* (1 – 1/(1.134*1.225*1.368*1.826)) = 73, 617 [2]

As these are claims paid, we don't need to calculate the revised ultimate liability to get the reserve, we can just total up the emerging liabilities:

(9, 249 + 23, 989 + 45, 458 + 73, 617) = 152, 313 [1]

The assumptions are:

- Payments from each origin year will develop in the same way
- Weighted average past inflation will be repeated in the future

[1]

- The first year is fully run-off

The estimated loss ratio is appropriate

[8]
[10 Marks]

Solution 9:

i) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch. [0.5]

An arbitrage opportunity means that:

(a) we can start at time 0 with a portfolio that has a net value of zero (implying that we are long in some assets and short in others). This is usually called a zero-cost portfolio. [0.5]

(b) at some future time T:

- the probability of a loss is 0
- the probability that we make a strictly positive profit is greater than 0. [1]

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired. **[2]**

ii) **Law of one price**

The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk- free) profit. [2]

iii) **a)** Using Put- Call parity, the value of put option should be:

$$p_t=c_t+ K\exp(-r(T-t))-S_t\exp(-q(T-t))$$

$$= 30+120\exp(-.05*.25)-125\exp(-.15*.25)$$
$$= 28.11$$ [2]

b) Arbitrage profit

If the put options are only Rs. 23 then they are cheap. If things are cheap then we buy them.

So looking at the put-call parity relationship, we "buy the cheap side and sell the expensive side", *ie* we buy put options and shares and sell call options and cash. [1]

For example:

- sell 1 call option Rs. 30
- buy 1 put option (Rs. 23)
- buy 1 share (Rs.125)
- sell (borrow) cash Rs.118

This is a zero-cost portfolio and, because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows: [1]

In 3 months' time, repaying the cash will cost us:
118exp(0.05*3 /12)= Rs. 119.48

We also receive dividends *d* on the share.

1. If the share price is above 120 in 3 months' time then the other party will exercise their call option and we will have to give them the share. They will pay 120 for it and our profit is:

$$120 - 119.48 + d = 0.52 + d$$

(The put option is useless to us) [1]

2. If the share price is below 120 in 3 months' time then we will exercise our put option and sell it for 120. Our profit is:

$$120 - 119.48 + d = 0.52 + d$$

(The call option is useless to the other party and will expire worthless) [1]

[4]

[10 Marks]

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